Stability Analysis of Linear Continuous-Time Fractional Systems of Commensurate Order

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Abstract:

The paper considers the stability problem of linear time-invariant continuous-time systems of fractional commensurate order. It is shown that the system is stable if and only if plot of rational function of fractional order (called the generalised modified Mikhailov plot) does not encircle the origin of the complex plane. The considerations are illustrated by numerical examples.

Keywords: *stability, linear system, continuous-time, fractional, commensurate order.*

1. Introduction

In the last decades, the problem of analysis and synthesis of dynamical systems described by fractional order differential (or difference) equations was considered in many papers, see [4, 9, 10, 12, 18, 21, 23], for example. Some applications of fractional order systems can be found in [5, 19, 20-22]. The stability problem of linear continuous-time systems of fractional order has been studied in [2, 3, 6, 7, 8, 23]. The new class of the linear fractional order systems, namely the positive systems of fractional order has been considered in [15-17].

The aim of this paper is to give the new frequency domain methods for stability analysis of linear continuous-time fractional systems in the case of commensurate degree characteristic polynomials. To the best knowledge of the Author, computationally effective frequency domain methods for stability analysis of fractional commensurate degree polynomials have not been proposed yet.

2. Preliminaries and problem formulation

A linear single input, single output continuous-time dynamical system of fractional order is described by the following fractional differential equation (see [23] for example)

$$\sum_{i=0}^{n} a_i \frac{d^{\alpha_i}}{dt^{\alpha_i}} y(t) = \sum_{k=0}^{m} b_k \frac{d^{\beta_k}}{dt^{\beta_k}} u(t), \tag{1}$$

where $u(t)$ is the input, $y(t)$ is the output, $\alpha_n > \alpha_{n-1} > \ldots > \alpha_1 > \alpha_0 \geq 0$ and $\beta_m > \beta_{m-1} > \ldots > \beta_1 > \beta_0 \geq 0$ are arbitrary real numbers, a_i $(i = 0,1,2,\ldots,n)$ and b_k $(k = 0,1,2,\ldots,m)$ are real coefficients and

$$\frac{d^{\alpha} x(t)}{dt^{\alpha}} = \frac{1}{\Gamma(p-\alpha)} \int_0^t \frac{x^{(p)}(\tau)d\tau}{(t-\tau)^{\alpha+1-p}} \tag{2}$$

is the Caputo definition for fractional α-order derivative where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$ is the Euler gamma function and p is an integer satisfying inequality $p-1 \leq \alpha \leq p$.

Applying the Laplace transform to both sides of equation (1) and assuming zero initial conditions, we obtain the following fractional order transfer function

$$G(s) = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \ldots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \ldots + a_0 s^{\alpha_0}}. \tag{3}$$

The fractional order linear system with the transfer function (3) is of [23]:

- commensurate order if

$$\alpha_i = i\alpha, \; i = 0,1,\ldots,n, \; \beta_k = k\alpha, \; k = 0,1,2,\ldots,m, \tag{4}$$

where $\alpha > 0$ is a real number,

- rational order if it is a commensurate order and $\alpha = 1/q$ where q is a positive integer,
- non-commensurate order if (4) does not hold.

The transfer function of fractional system of commensurate order can be written in the form

$$G(s) = \frac{b_m s^{m\alpha} + b_{m-1} s^{(m-1)\alpha} + \ldots + b_0}{a_n s^{n\alpha} + a_{n-1} s^{(n-1)\alpha} + \ldots + a_0}. \tag{5}$$

Substituting $\lambda = s^{\alpha}$ in (5), one obtains the associated natural order transfer function

$$\widetilde{G}(\lambda) = \frac{b_m \lambda^m + b_{m-1} \lambda^{m-1} + \ldots + b_0}{a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_0}. \tag{6}$$

If, for example,

$$G(s) = \frac{s^{0.25} + 1}{s - 2s^{0.5} + 1.25}, \tag{7a}$$

then for $\lambda = s^{0.25}$ one obtains the associated natural order transfer function

$$\widetilde{G}(\lambda) = \frac{\lambda + 1}{s^4 - 2\lambda^2 + 1.25}. \tag{7b}$$

Characteristic polynomial of the fractional system (1) has the form

$$D(s) = a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \ldots + a_0 s^{\alpha_0}. \tag{8}$$

The polynomial (8) is a multivalued function whose domain is a Riemann surface. In general, this surface has an infinite number of sheets and the fractional polynomial (8) has an infinite number of zeros. Only a finite number of which will be in the main sheet of the Riemann surface. For stability reasons only the main sheet defined by $-\pi < \arg s < \pi$ can be considered [23].

Theorem 1 [7, 8, 23]. The fractional order system with the transfer function (3) is bounded-input boundedoutput (BIBO) stable (shortly stable) if and only if the fractional degree characteristic polynomial (8) is stable, i.e. this polynomial has no zeros in the closed right half of the Riemann complex surface, that is

$$D(s) \neq 0 \text{ for } \mathrm{Re}\, s \geq 0. \tag{9}$$

The Riemann surface has a finite number of sheets only in the case of fractional polynomials (8) of commensurate degree, i.e. for

$$\alpha_i = i\alpha, \ \ i = 0,1,\ldots,n. \tag{10}$$

If (10) holds, the fractional degree characteristic polynomial (8) can be written in the form

$$D(s) = a_n s^{n\alpha} + a_{n-1}s^{(n-1)\alpha} + \ldots + a_0. \tag{11}$$

Hence, for $\lambda = s^{\alpha}$ from (11) we obtain the associated natural number degree polynomial

$$\widetilde{D}(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0. \tag{12}$$

If, for example, $D(s) = s^{1.5} + as^{0.5} + b$ (a and b are real numbers) then , $\alpha = 1/2$, $\lambda = s^{1/2}$ and the associated natural number degree polynomial has the form $\widetilde{D}(\lambda) = \lambda^3 + a\lambda + b.$

Theorem 2 [23]. The fractional commensurate degree characteristic polynomial (11) is stable if and only if all zeros of this polynomial satisfy the condition (9) or, equivalently, all zeros λ_i of the associated natural degree polynomial (12) satisfy the condition

$$|\arg\lambda_i| > \alpha\frac{\pi}{2}, \ \ i = 1,2,\ldots,n. \tag{13}$$

If $0 < \alpha \leq 1$ then from (13) we obtain the stability region shown in Figure 1.

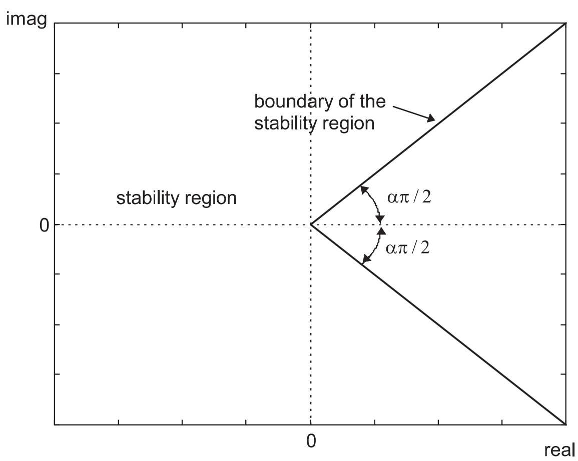

Fig. 1. Stability region of fractional degree polynomial (11) in the complex $\lambda = s^{\alpha}$-plane with $0 < \alpha \leq 1$.

Parametric description of the boundary of the stability region has the form

$$(j\omega)^{\alpha} = |\omega|^{\alpha} e^{j\pi\alpha/2}, \ \ \omega \in (-\infty, \infty). \tag{14}$$

The polynomial (11) with $\alpha = 1$ is a natural number degree polynomial and from (14) we have that the imaginary axis of the complex plane is the boundary of the stability region.

From the above and Theorem 2 we have the following sufficient condition for stability of fractional degree polynomial (11) with $0 < \alpha \leq 1$.

Lemma 1. The fractional commensurate degree characteristic polynomial (11) with $0 < \alpha \leq 1$ is stable if the associated natural number degree polynomial (12) is asymptotically stable, i.e. the condition (13) holds for $\alpha = 1$, which means that $|\arg\lambda_i| > \pi/2$ for all zeros λ_i of (12).

From Theorem 2 it follows that the fractional polynomial (11) may be stable in the case when the associated natural degree polynomial (12) is not asymptotically stable.

Stability checking of the fractional degree polynomial (11) on the basis of Theorem 2 is a difficult problem in general, because the degree of the associated polynomial (12) may be very large. If, for example,

$$D(s) = s^{127/105} + 0.4s^{77/105} + 0.3s^{71/105} + 0.1s^{56/105} + 1,$$

then for $\lambda = s^{\alpha} = s^{1/105}$ one obtains the associated polynomial of natural degree [6]

$$\widetilde{D}(\lambda) = \lambda^{127} + 0.4\lambda^{77} + 0.3\lambda^{71} + 0.1\lambda^{56} + 1.$$

The above polynomial has degree equal to 127 and only five non-zero coefficients.

To avoid this difficulty, a method for determination of the multi-variate natural degree polynomial, associated with the fractional commensurate degree polynomial has been given in [6]. To stability analysis of multi-variate degree polynomials, the LMI technique has been proposed in [6].

The aim of this paper is to give the new frequency domain methods for stability analysis of fractional polynomials of commensurate degree. The methods proposed are based on the Mikhailov stability criterion and the modified Mikhailov stability criterion, known from the theory of systems of natural number order (see [1, 14, 24], for example).

3. Solution of the problem

In the stability theory of natural degree characteristic polynomials of linear continuous-time systems, the following kinds of stability are considered (see [1], for example):

- asymptotic stability (all zeros of the characteristic polynomial have negative real parts),
- D-stability (all zeros of the characteristic polynomial lie in the open region D in the left half-plane of complex plane).

From the above and Theorem 2 we have the following lemma.

Lemma 2. The fractional degree polynomial (11) is stable if and only if the associated natural degree polynomial (12) is D-stable, where the parametric description the boundary of the region D has the form (14). In particular, for $0 < \alpha \leq 1$ the D-stability region is shown in Figure 1.

It is easy to see that if $\alpha = 1$ then the fractional degree polynomial (11) is reduced to the natural degree polynomial (12) with $\lambda = s$. In such a case from (14) it follows that boundary of the stability region is the imaginary axis of the complex plane.

Theorem 3. The fractional degree characteristic polynomial (11) is stable if and only if

$$\underset{0 \leq \omega < \infty}{\Delta \arg} D(j\omega) = n\pi / 2, \tag{15}$$

where $D(j\omega) = D(s)$ for $s = j\omega$.

Proof. It is easy to see that $\widetilde{D}((j\omega)^{\alpha}) = D(j\omega)$. This means that (15) is the necessary and sufficient condition for D-stability of the natural degree polynomial (12) [1]. Hence, the proof follows from Lemma 2. ■

Plot of the function $D(j\omega)$, where $D(j\omega) = D(s)$ for $s = j\omega$ will be called the generalised (to the class of fractional degree polynomials) Mikhailov plot.

Satisfaction of (15) means that the generalised Mikhailov plot starts for $\omega = 0$ in the point $D(j0)$ on real axis and with ω increasing from 0 to ∞ turns strictly counter-clockwise and goes through n quadrants of the complex plane.

Checking the condition (15) of Theorem 3 is difficult in general (for large values of n), because $D(j\omega)$ quickly tends to infinity as ω grows to ∞.

To remove this difficulty, we consider the rational function

$$\psi(s) = \frac{D(s)}{w_r(s)}, \tag{16}$$

instead of the polynomial (11), where $w_r(s)$ is the reference fractional polynomial of the same degree as polynomial (11).

We will assume that the reference fractional polynomial $w_r(s)$ is stable, i.e.

$$w_r(s) \neq 0 \text{ for } \operatorname{Re} s \geq 0. \tag{17}$$

Theorem 4. The fractional degree polynomial (11) is stable if and only if

$$\underset{\omega \in (-\infty, \infty)}{\Delta \arg} \psi(j\omega) = 0, \tag{18}$$

where $\psi(j\omega) = \psi(s)$ for $s = j\omega$ and $\psi(s)$ is defined by (16).

Proof. If the reference polynomial $w_r(s)$ is stable then from Theorem 3 we have

$$\underset{\omega \in (-\infty, \infty)}{\Delta \arg} w_r(j\omega) = n\pi. \tag{19}$$

From (16) for $s = j\omega$ it follows that

$$\Delta \arg \psi(j\omega) = \Delta \arg D(j\omega) - \Delta \arg w_r(j\omega). \tag{20}$$

The fractional degree polynomial (11) is stable if and only if

$$\underset{\omega \in (-\infty, \infty)}{\Delta \arg} D(j\omega) = \underset{\omega \in (-\infty, \infty)}{\Delta \arg} w_r(j\omega) = n\pi,$$

which holds if and only if (18) is satisfied. ■

The reference fractional polynomial $w_r(s)$ can be chosen in the form

$$w_r(s) = a_n (s + c)^{\alpha n}, \ c > 0. \tag{21}$$

Note that for $c > 0$ the reference polynomial (21) is stable.

Plot of the function $\psi(j\omega)$, $\omega \in (-\infty, \infty)$ ($\psi(s)$ is defined by (16)) we will called the generalised modified Mikhailov plot.

The condition (18) of Theorem 4 holds if and only if the generalised modified Mikhailov plot does not encircle the origin of the complex plane as ω runs from $-\infty$ to ∞.

From (11), (16) and (21) we have

$$\lim_{\omega \to \pm\infty} \psi(j\omega) = \lim_{\omega \to \pm\infty} \frac{D(j\omega)}{w_r(j\omega)} = 1, \tag{22}$$

and

$$\psi(j0) = \frac{w(j0)}{w_r(j0)} = \frac{a_0}{a_n c^{\alpha n}}. \tag{23}$$

From (23) it follows that $\psi(j0) \leq 0$ if $a_0 / a_n \leq 0$ Hence, from Theorem 4 we have the following important lemma.

Lemma 3. The fractional degree polynomial (11) is not stable if $a_0 / a_n \leq 0$.

Now we consider the case in which the condition (18) of Theorem 4 does not hold.

Theorem 5. The fractional characteristic polynomial (11) of commensurate degree has $k \geq 0$ zeros in the right half of the Riemann complex surface if and only if as ω runs from $-\infty$ to $+\infty$ the plot of $\psi(j\omega)$ k times encirclese in the negative direction the origin of the complex plane. In such a case

$$\underset{\omega \in (-\infty, \infty)}{\Delta \arg} \psi(j\omega) = -k2\pi. \tag{24}$$

Proof. As in [14] in the case of natural degree polynomials we can show that if the fractional degree polynomial (11) has $k \geq 0$ zeros with positive real parts, then

$$\underset{\omega \in (-\infty, \infty)}{\Delta \arg} D(j\omega) = (n - 2k)\pi. \tag{25}$$

Hence, from (20) and (19), (25) it follows that (24) holds. If (24) holds then from (20) and (19) we have (25). ■

It is easy to see that Theorem 4 follows from Theorem 5 for $k = 0$.

4. Illustrative examples

Example 1.

Consider a linear fractional order system with characteristic polynomial of commensurate degree of the form [6]

$$
\begin{aligned}
D(s) = {} & 134.7955988s^{11/15} + 17.49138877s^{14/15} + \\
& + 7.5619s^{7/5} + 18.60416827s^{6/5} + s^{8/5} + \\
& + 13.68686363s^{3/5} + 276.0731421s^{1/3} + \\
& + 269.6615050s^{1/5} + 218.5809037s^{2/5} + \\
& + 338.6269398s^{8/15} + 7.3225s^{19/15} + \\
& + 55.92198403s^{16/15} + 139.1374509s^{13/15} + \\
& + 14.79208246s + 221.9590294.
\end{aligned} \tag{26}
$$

For $\alpha = 1/15$ and $\lambda = s^{\alpha} = s^{1/15}$ from the fractional commensurate degree polynomial (26) we obtain the associated natural degree polynomial

$$
\begin{aligned}
\widetilde{D}(\lambda) = {} & \lambda^{24} + 7.5619\lambda^{21} + 7.3225\lambda^{19} + \\
& + 18.60416827\lambda^{18} + 55.92198403\lambda^{16} + \\
& + 14.79208246\lambda^{15} + 17.49138877\lambda^{14} + \\
& + 139.1374509\lambda^{13} + 134.7955988\lambda^{11} + \\
& + 13.68686363\lambda^{9} + 338.6269398\lambda^{8} + \\
& + 218.5809037\lambda^{6} + 276.0731421\lambda^{5} + \\
& + 269.6615050\lambda^{3} + 221.9590294.
\end{aligned} \tag{27}
$$

From Theorem 2 it follows that the fractional degree polynomial (26) is stable if and only if the associated natural degree polynomial (27) has no zeros in the cone shown in Figure 1 with $\alpha\pi/2 = \pi/30 = 0.1047$ rad.

Plot of the function (16) with $w_r(s) = (s+10)^{8/5}$ is shown in Figure 2. According to (22) and (23) we have

$$
\lim_{\omega \to \pm\infty} \psi(j\omega) = 1,
$$

$$
\psi(j0) = a_0 / 10^{8/5} = 221.9590294 / 10^{8/5} = = 5.5754
$$

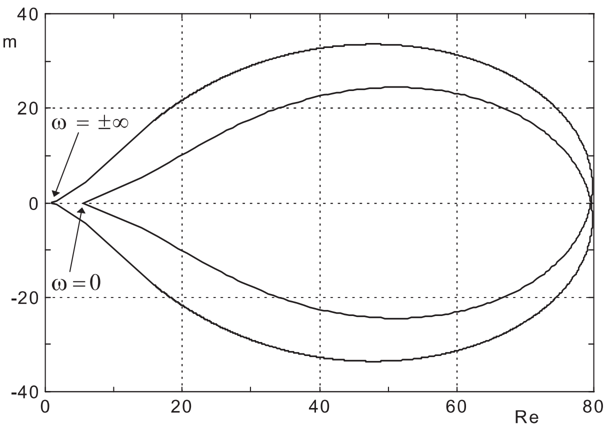

Fig. 2. Plot of the function (16) with $s = j\omega$, $\omega \in (-\infty, \infty)$.

From Figure 2 it follows that the generalised modified Mikhailov plot $\psi(j\omega)$ does not encircle the origin of the complex plane and the system is stable, according to Theorem 4.

Now we consider the fractional degree polynomial (26) and associated natural degree polynomial (27) in the case when the free term has negative sign, i.e. is -221.9590294 instead of +221.9590294. In such a case $a_0 / a_n = a_0 = -221.9590294 < 0$ and the fractional system is not stable, according to Lemma 3.

In this case, the generalised modified Mikhailov plot with the reference polynomial $w_r(s) = (s+10)^{8/5}$ is shown in Figure 3, where

$$
\lim_{\omega \to \pm\infty} \psi(j\omega) = 1, \quad \psi(j0) = -5.5754
$$

Zeros of natural degree polynomial (27) with negative free term and the boundary of the stability region are shown in Figure 4.

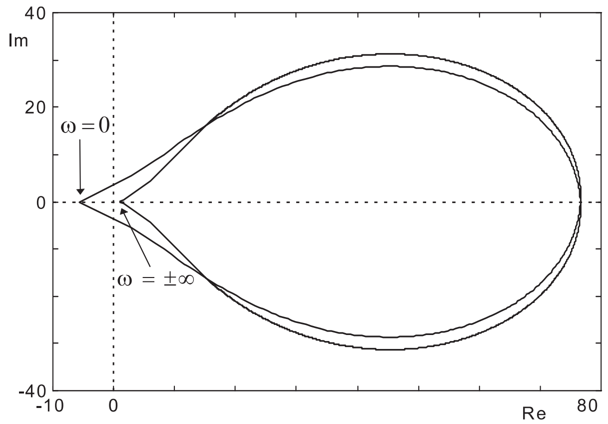

Fig. 3. Plot of the function (16) with $s = j\omega$, $\omega \in (-\infty, \infty)$ $D(s)$ of the form (26) with negative free term.

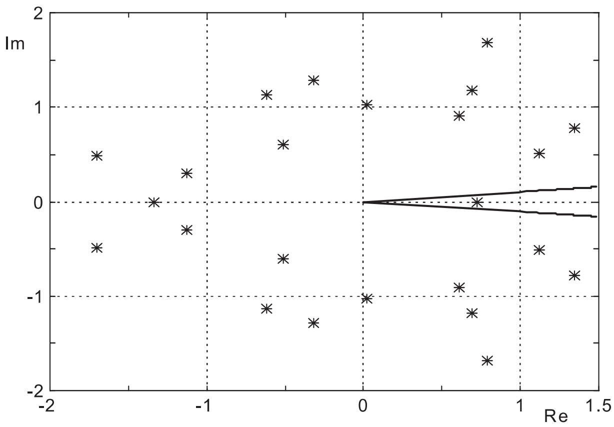

Fig. 4. Zeros of polynomial (27) with negative free term and boundary of the stability region.

From Figure 3 it follows that the generalised modified Mikhailov plot $\psi(j\omega)$ ones encircles the origin of the complex plane in negative direction. This means, according to Theorem 5, that the system is unstable and the characteristic polynomial has one unstable zero. This zeros lies in the instability region shown in Figure 4.

Example 2. Consider the control system with the fractional order plant described by the transfer function [11, 25]

$$
G_0(s) = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1} = \frac{1}{D_0(s)} \tag{28}
$$

and the fractional PD controller (designed in [11])

$$
C(s) = 20.5 + 3.7343s^{1.15}. \tag{29}
$$

Characteristic polynomial of the closed loop system with the plant (28) and controller (29) has the form

$$D(s) = D_0(s) + C(s) = \\ = 0.8s^{2.2} + 3.7343s^{1.15} + 0.5s^{0.9} + 21.5\,. \tag{30}$$

Substituting $\alpha = 1/20$ and $\lambda = s^{\alpha} = s^{1/20}$ in (30), one obtains the associated polynomial of natural degree

$$\widetilde{D}(\lambda) = 0.8\lambda^{44} + 3.7343\lambda^{23} + 0.5\lambda^{18} + 21.5\,. \tag{31}$$

The control system is stable if and only if all zeros of polynomial (31) lie in the stability region shown in Figure 1 with $\alpha = 1/20$.

To stability checking of fractional polynomial (30) we apply Theorem 4.

Plot of the function $\psi(j\omega) = D(j\omega)/w_r(j\omega)$ where $D(s)$ has the form (30) and $w_r(s) = 0.8(s+5)^{2.2}$ is the reference fractional polynomial, is shown in Figure 5.

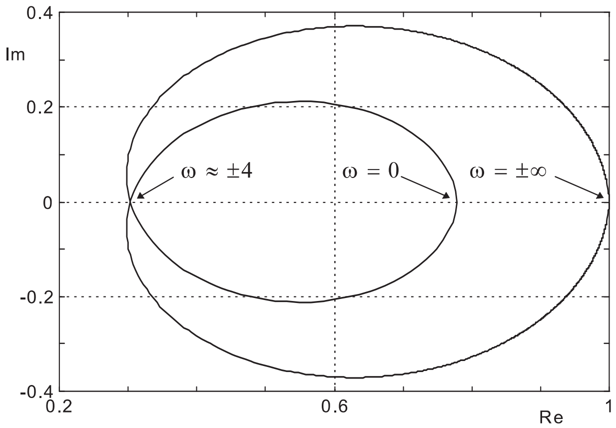

Fig. 5. Plot of the function $\psi(j\omega)$.

From (22) and (23) we have

$$\lim_{\omega \to \pm\infty} \psi(j\omega) = 1; \;\; \psi(j0) = \frac{D(j0)}{w_r(j0)} = \\ = \frac{21.5}{0.8 \cdot 5^{2.2}} = 0.7791.$$

From Figure 5 it follows that the generalised modified Mikhailov plot $\psi(j\omega)$ does not encircle the origin of the complex plane. This means that the fractional control system is stable, according to Theorem 4.

5. Concluding remarks

New frequency domain methods for stability analysis of linear systems of fractional commensurate order have been given.

The methods have been obtained by generalisation of the Mikhailov stability criterion and the modified Mikhailov stability criterion (known from the theory of natural order systems) to the case of fractional order systems.

In particular it has been shown that:

- the fractional polynomial (11) is stable if and only if plot of $D(j\omega)$ with ω increasing from 0 to ∞ turns strictly counter-clockwise and goes through n quadrants of the complex plane (Theorem 3),
- the fractional polynomial (11) is stable if and only if plot of the rational function $\psi(j\omega)$, $\omega \in (-\infty, \infty)$ where $\psi(s)$ is defined by (16), called the generalised modified Mikhailov plot, does not encircle the origin of the complex plane (Theorem 4),
- the fractional characteristic polynomial (11) has $k \geq 0$ zeros in the right half of the Riemann complex surface if and only if as ω runs from $-\infty$ to $+\infty$ the generalised modified Mikhailov plot $\psi(j\omega)$ k times encircles in the negative direction the origin of the complex plane (Theorem 5).

The effectiveness of the methods has been illustrated by numerical examples.

Generalisation of the main result of the paper (Theorem 4) for the fractional systems of non-commensurate order has been given in [2].

The preliminary version of this paper was presented at the conference Automation'2008 which was held in Warsaw, Poland and published in [3].

The considerations can be generalised to the linear fractional order systems with delays.

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References

[1] Busłowicz M., *Stability of linear time-invariant systems with uncertain parameters*, Publishing Department of Technical University of Białystok, Białystok 1997 (in Polish).

[2] Busłowicz M., "Frequency domain method for stability analysis of linear continuous-time fractional systems". In: K. Malinowski, L. Rutkowski (Eds.): *Recent Advances in Control and Automation,* Academic Publishing House EXIT: Warsaw 2008, pp. 83-92.

[3] Busłowicz M., "Stability of linear continuous-time fractional systems of commensurate order", *Pomiary Automatyka Robotyka*, no. 2, 2008, 475-484 (on CD-ROM) (in Polish).

[4] Das. S, *Functional Fractional Calculus for System Identification and Controls*, Springer, Berlin 2008.

[5] Ferreira N. M. F., Machado J. A. T., "Fractional-order hybrid control of robotic manipulators". In: *Proc. of 11th Int. Conf. Advanced Robotics*, ICAR'2003, Coimbra 2003, Portugal, 393-398.

[6] Gałkowski K., Bachelier O., Kummert A., "Fractional polynomial and nD systems a continuous case. In: *Proc. of IEEE Conference on Decision & Control*, San Diego 2006, USA.

[7] Matignon D., "Stability results on fractional differential equation with applications to control processing". In: *Proc. of IMACS*, Lille 1996, France.

[8] Matignon D., "Stability properties for generalized fractional differential systems". In: *Proc. of ESAIM*, 1998, pp. 145-158.

[9] Ortigueira M. D., "Introduction to fractional linear systems. Part 1: Continuous-time case". In: *IEE Proc. – Vis.*

Image Signal Process, no. 147, 2000, pp. 62-70.
[10] Ortigueira M. D., "Introduction to fractional linear systems. Part 2: Discrete-time systems". In: *IEE Proc. – Vis. Image Signal Process*, no. 147, 2000, pp. 71-78.
[11] Podlubny I., *Fractional order systems and fractional order controllers.* The Academy of Sciences Institute of Experimental Physis, Kosice, Slovak Republic, 1994.
[12] Podlubny I., Fractional Differential Equations, Academic Press, San Diego 1999.
[13] Sierociuk D., *Estimation and control of discrete dynamical systems of fractional order in state space*, PhD Dissertation, Faculty of Electrical Engineering, Warsaw University of Technology, Warsaw 2007 (in Polish).
[14] Kaczorek T., *Theory of Control Systems*, WNT: Warszawa, 1974 (in Polish).
[15] Kaczorek T., "Reachability and controllability to zero of positive fractional discrete-time systems", *Machine Intelligence and Robotics Control*, vol. 6, no. 4, 2008, pp. 139-143.
[16] Kaczorek T. "Reachability of fractional positive continuous-time linear systems", *Pomiary Automatyka Robotyka*, no. 2, 2008, pp. 527-537 (on CD-ROM).
[17] Kaczorek T, "Fractional positive continuous-time linear systems and their reachability", *Int. J. Appl. Math. Comput. Sci.*, vol. 18, no. 2, 2008, pp. 223-228.
[18] Kilbas A. A., Srivastava H. M. Trujillo J. J., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam 2006.
[19] Reyes-Melo E., Martinez-Vega J.J., Guerrero-Salazar C.A., Ortiz-Mendez U., "Modelling and relaxation phenomena in organic dielectric materials. Application of differential and integral operators of fractional order". *J. Optoel. Adv. Mat.*, vol. 6, no , 2004, pp. 1037-1043.
[20] Riu D., Retiére N., Ivanes M., "Turbine generator modeling by non-integer order systems". In: *Proc. IEEE Int. Electric Machines and Drives Conference*, IEMDC 2001, England, Cambridge, 2001, pp. 185-187.
[21] Sabatier J., Agrawal O. P., Machado J. A. T. (Eds), *Advances in Fractional Calculus, Theoretical Developments and Applications in Physics and Engineering*, Springer, London 2007.
[22] Sjöberg M., Kari L., "Non-linear behavior of a rubber isolator system using fractional derivatives". V*ehicle Syst. Dynam.*, vol. 37, no. 3, 2002, pp. 217-236.
[23] Vinagre B. M., Monje C. A., Calderon A. J., "Fractional order systems and fractional order control actions", Lecture 3. In: *Proc. of IEEE CDC'02 TW#2: Fractional Calculus Applications in Automatic Control and Robotics*, Las Vegas 2002.
[24] Wright W. C., Kerlin T. W., "An efficient computer oriented method for stability analysis of large multivariable systems", *Trans. ASME Journal Basic Eng.*, no. 92, 1970, pp. 279-286.
[25] Zhao Ch., Xue D., Chen Y.-Q. (2005), "A fractional order PID tuning algorithm for a class of fractional order plants". In: *Proc. IEEE Intern. Conf. on Mechatronics & Automation*, Niagara Falls 2005, Canada, pp. 216-221.