POINTWISE COMPLETENESS AND POINTWISE DEGENERACY OF LINEAR CONTINUOUS-TIME FRACTIONAL ORDER SYSTEMS

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Abstract:

A dynamical system described by homogeneous equation is called pointwise complete if every final state can be reached by suitable choice of the initial state. The system, which is not pointwise complete, is called pointwise degenerated. Definitions and necessary and sufficient conditions for the pointwise completeness and the pointwise degeneracy of continuous-time linear systems of fractional order, standard and positive, are given. It is shown that: 1) the standard fractional system is always pointwise complete; 2) the positive fractional system is pointwise complete if and only if the state matrix is diagonal.

Keywords: *inear system, fractional, continuous-time, positive, pointwise completeness.*

1. Introduction

A dynamical system without input signal is called pointwise complete if every final state can be reached by suitable choice of the initial state. The system, which is not pointwise complete, is called pointwise degenerated.

The problem of pointwise completeness and pointwise degeneracy of linear continuous-time systems with delays has been considered in [5, 14, 16, 18].

The problem of pointwise completeness and pointwise degeneracy of linear discrete-time systems with delays has been formulated and solved in [1, 2] for the standard systems and in [4] for the positive systems.

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive systems is given in the monographs [7, 10].

In the last decades, the problem of analysis and synthesis of dynamical systems described by fractional order differential (or difference) equations was considered in many papers and books (see [6, 15, 17], for example).

The new class of linear systems of fractional order, namely the positive fractional systems, has been introduced in [11-13].

In this paper we consider the problem of pointwise completeness and pointwise degeneracy of linear continuous-time systems of fractional order. Definitions and necessary and sufficient conditions for the pointwise completeness and the pointwise degeneracy of fractional systems standard and positive will be given.

To the best knowledge of the authors the pointwise completeness and pointwise degeneracy of fractional order systems have not been considered yet.

In the paper the following notation will be used: $\Re^{n\times m}$ - the set of $n\times m$ real matrices and $\Re^n=\Re^{n\times 1}$; $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with non-negative entries and $\Re_+^n=\Re_+^{n\times 1}$; I_n - the $n\times n$ identity matrix.

2. The main results

2.1. Standard systems

Consider the fractional continuous-time linear system described by the homogeneous equation

$$\frac{d^\alpha x(t)}{dt^\alpha}=Ax(t), \tag{1}$$

where $0<\alpha<1$ is the order of fractional derivative, $x=x(t)\in\Re^n$ and $A\in\Re^{n\times n}$.

The following Caputo definition of the fractional α-order derivative will be used [12]

$$\frac{d^\alpha x(t)}{dt^\alpha}=\frac{1}{\Gamma(p-\alpha)}\int_0^t\frac{x^{(p)}(\tau)d\tau}{(t-\tau)^{\alpha+1-p}} \tag{2}$$

where $\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt$ is the Euler gamma function and p is an integer satisfying the inequality $p-1\le\alpha\le p$.

It is easy to see that for $0<\alpha<1$ we have $p=1$ and

$$\frac{d^\alpha x(t)}{dt^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{x^{(1)}(\tau)d\tau}{(t-\tau)^\alpha}. \tag{2a}$$

The solution of equation (1) with $x(0)=x_0$ is given by [12]

$$x(t)=\Phi_0(t)x_0, \tag{3}$$

where

$$\Phi_0(t)=\Phi_0(A,t)=E_\alpha(At^\alpha)=\sum_{k=0}^\infty\frac{A^kt^{k\alpha}}{\Gamma(k\alpha+1)} \tag{4}$$

is the fundamental matrix and $E_\alpha(At^\alpha)$ is the Mittage-Leffler matrix function.

The fundamental matrix $\Phi_0(t)$ depends on the time t and the matrix A.

Lemma 1. The fundamental matrix (4) is always nonsingular, i.e.

$$\det\Phi_0(A,t)\neq 0 \tag{5}$$

for all $t\ge 0$ and for any matrix $A\in\Re^{n\times n}$.

Proof. Let us consider the function

$$\Phi_0(z,t) = \sum_{k=0}^{\infty} \frac{z^k t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{6}$$

We show that $\det \Phi_0(A,t) \neq 0$ for any matrix $A \in \Re^{n\times n}$.

Note that the function (6) is well definite on the spectrum of the matrix A. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be real or complex eigenvalues (not necessarily distinct) of A. Then from (6) it follows that $\Phi_0(\lambda_i, t) \neq 0$ for any real λ_i $(i = 1,2,...,n)$ and $\Phi_0(\lambda_i,t)\Phi_0(\lambda_{i+1},t) \neq 0$ for any complex conjugate pair (λ_i, λ_{i+1}), $i \in (1,2,...,n-1)$.

It is well known [8, 9] that the eigenvalues of the matrix $\Phi_0(A,t)$ are: $\Phi_0(\lambda_1,t), \Phi_0(\lambda_2,t)$, ..., $\Phi_0(\lambda_n,t)$ and

$$\det \Phi_0(A,t) = \Phi_0(\lambda_1,t)\Phi_0(\lambda_2,t)\cdots\Phi_0(\lambda_n,t) \neq 0. \tag{7}$$

■

The fundamental matrix (4) can be computed by using the Sylvester formula [8, 9]. In the case of distinct eigenvalues of A we have the following lemma.

Lemma 2. If A has only distinct eigenvalues then the fundamental matrix (4) can be computed from the formula

$$\Phi_0(A,t) = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{A - I_n\lambda_j}{\lambda_i - \lambda_j} \Phi_0(\lambda_i,t). \tag{8}$$

Example 1. Using the Sylvester formula check the fundamental matrix of the system (1) with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{9}$$

The matrix (9) has two distinct eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 0$. In this case, according to Sylvester formula (8), we have

$$\Phi_0(A,t) = A\Phi_0(\lambda_1,t) + (I_2 - A)\Phi_0(\lambda_2,t) = \\ = \begin{bmatrix} \varphi(t) & 0 \\ 0 & 1 \end{bmatrix}, \tag{10}$$

where

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{11}$$

■

From (4) it follows that the fundamental matrix can be written in the form

$$\Phi_0(t) = I_n + \sum_{k=1}^{\infty} A^k \varphi_k(t), \tag{12}$$

where

$$\varphi_k(t) = \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{13}$$

From (12) we have the following lemma.

Lemma 3. If A is a nilpotent matrix with the nilpotency index μ (i.e. $A^k = 0$ for $k = \mu, \mu+1, \ldots$ and $A^{\mu-1} \neq 0$) then

$$\Phi_0(t) = I_n + \sum_{k=1}^{\mu-1} A^k \varphi_k(t). \tag{14}$$

By generalisation of definitions of pointwise completeness and pointwise degeneracy to the case of fractional order systems (1) one obtains the following definitions.

Definition 1. The fractional system (1) is called pointwise complete at $t = t_f$ if for every final vector $x_f \in \Re^n$ there exists an initial state $x_0 \in \Re^n$ such that $x(t_f) = x_f$.

Definition 2. The fractional system (1) is called pointwise degenerated in the direction v at $t = t_f$ if there exists a non-zero vector $v \in \Re^n$ such that for all initial states $x_0 \in \Re^n$ the solution of (1) for $t = t_f$ satisfies the condition $v^{\mathrm{T}} x_f = 0$, where T denotes the transpose.

From the above definitions and Lemma 1 we have the following important theorem.

Theorem 1. The fractional continuous-time system (1) is always pointwise complete, i.e. for any finite state x_f there exists the initial state

$$x_0 = \Phi_0^{-1}(t_f)x_f \tag{15}$$

such that $x(t_f) = x_f$.

Proof. From Lemma 1 we have that $\det \Phi_0(t) \neq 0$ for any matrix $A \in \Re^{n\times n}$ and for all $t \geq 0$. Hence, from (3) it follows that for any given finite state x_f we can always compute the initial state from the formula (15). ■

Example 2. Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{16}$$

It is easy to see that (16) is a nilpotent matrix with the nilpotency index $\mu = 2$ i.e.

$$A^k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ for } k = 2,3,\ldots.$$

From Lemma 3 for $\mu = 2$ we have

$$\Phi_0(t) = I_2 + A\varphi_1(t) = \begin{bmatrix} 1 & \varphi_1(t) \\ 0 & 1 \end{bmatrix}, \tag{17}$$

where $\varphi_1(t)$ is defined by (13) for $k = 1$.

From Theorem 1 it follows that the fractional system is pointwise complete and for any final state $x_f = [x_{f1} \quad x_{f2}]^{\mathrm{T}} \in \Re^2$ we can find the suitable initial state from the formula

$$x_0 = \Phi_0^{-1}(t_f)x_f = \\ = \begin{bmatrix} 1 & -\varphi_1(t_f) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix} = \begin{bmatrix} x_{f1} - x_{f2}\varphi_1(t_f) \\ x_{f2} \end{bmatrix}. \tag{18}$$

If, for example, $x_{f1} = 0$ and $x_{f2} > 0$ then

$$x_0 = \begin{bmatrix} -x_{f2}\varphi_1(t_f) \\ x_{f2} \end{bmatrix}. \tag{19}$$

2.2. Positive systems

Definition 3. The fractional system (1) is called positive if and only if $x(t) \in \Re_+^n$, $t \geq 0$ for any $x_0 \in \Re_+^n$.

A square real matrix $A = [a_{ij}]$ is called the Metzler matrix if its off-diagonal entries are non-negative, i.e. $a_{ij} \geq 0$ for $i \neq j$.

In the paper [12] the following theorem has been proved.

Theorem 2. The fractional system (1) is positive if and only if the matrix A is a Metzler matrix.

Definitions of pointwise completeness and pointwise degeneracy of positive fractional system (1) can be formulated as follows.

Definition 4. The positive fractional system (1) is called pointwise complete at $t = t_f$ if for every vector $x_f \in \Re_+^n$ there exists an initial state $x_0 \in \Re_+^n$ such that $x(t_f) = x_f$.

Definition 5. The positive fractional system (1) is called pointwise degenerated if it is not pointwise complete, that is there exists at least one state $x_f \in \Re_+^n$ which can not be reached from any initial state $x_0 \in \Re_+^n$ i.e. does not exist t_f and $x_0 \in \Re_+^n$ such that $x(t_f) = x_f$.

Theorem 3. The positive fractional system (1) is pointwise complete at $t = t_f$ if and only if the matrix A is diagonal.

Proof. In positive systems, according to Definition 3, $x_0 \in \Re_+^n$ and $x_f \in \Re_+^n$. From (15) it follows that for any $x_f \in \Re_+^n$ there exists $x_0 \in \Re_+^n$ if and only if $\Phi_0^{-1}(t_f) \in \Re_+^n$. It is well known [10] that $\Phi_0^{-1}(t_f) \in \Re_+^n$ if and only if $\Phi_0(t_f)$ is a monomial matrix (in each row and in each column only one entry is positive and remaining entries are zero). From (12) it follows that $\Phi_0(t_f)$ is a monomial matrix if and only if the matrix A is diagonal. ■

From Definition 5 and Theorem 3 we have the following theorem.

Theorem 4. The positive fractional system (1) is pointwise degenerated if and only if A is not a diagonal matrix.

Example 3. In Example 2 it was shown that the standard system (1) with the Metzler matrix (16) is pointwise complete.

From (19) it follows that if $x_{f1} = 0$ and $x_{f2} > 0$ then $x_0 \notin \Re_+^2$. This means that the system with the Metzler matrix (16), analysed as a positive system, is pointwise degenerated. This result also follows from Theorem 3.

Example 4. Consider the positive system (1) with the matrix (9). Fundamental matrix of the system has the form (10).

By Theorem 3 the positive fractional system is pointwise complete at any $t = t_f$ since the matrix (10) is diagonal. Using (15) we may find $x_0 \in \Re_+^2$ for any given $x_f \in \Re_+^2$.

If $x_f = [x_{f1} \quad x_{f2}]^{\mathrm{T}} \in \Re_+^2$ then from (15) we have

$$x_0 = \begin{bmatrix} 1/\varphi(t_f) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix} = \begin{bmatrix} x_{f1}/\varphi(t_f) \\ x_{f2} \end{bmatrix} \in \Re_+^2.$$

3. Concluding remarks

The paper considers the problem of pointwise completeness and pointwise degeneracy of linear continuous-time systems of fractional order, described by the homogeneous equation (1). First, the definitions of the pointwise completeness and pointwise degeneracy of the standard (i.e. non-positive) fractional systems have been introduced (Definitions 1 and 2) and it has been proved that the fractional continuous-time system (1) is always pointwise complete (Theorem 1). Next, the definitions (Definitions 4 and 5) and necessary and sufficient conditions of the pointwise completeness (Theorem 3) and pointwise degeneracy (Theorem 4) of the positive fractional systems have been given. It has been shown that the positive system (1) is pointwise complete if and only if the state matrix A is diagonal.

The considerations can be extended for the fractional discrete-time systems [3].

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References

[1] Busłowicz M., "Controllability of linear discrete-delay systems". In: *Proc. Intern. Conf. Functional Differential Systems and Related Topics*, Błażejewko, Poland, 1981, pp. 47-51.

[2] Busłowicz M., "On some properties of solution of state equation of discrete-time systems with delays", *Zesz. Nauk. Polit. Biał., Elektrotechnika*, vol. 1, 1983, pp. 17-29 (in Polish).

[3] Busłowicz M., "Pointwise completeness and pointwise degeneracy of linear discrete-time systems of fractional order", *Zesz. Nauk. Pol. Śląskiej, Automatyka,* no. 151, 2008, pp. 19-24 (in Polish).

[4] Busłowicz M., Kociszewski R., Trzasko W., "Pointwise completeness and pointwise degeneracy of positive discrete-time systems with delays", *Zesz. Nauk. Pol. Śląskiej, Automatyka*, no. 145, 2006, pp. 51-56 (in Polish).

[5] Choundhury A. K., "Necessary and sufficient conditions of pointwise completeness of linear time-invariant delay-differential systems", *Int. J. Control*, vol. 16 (6), 1972, pp. 1083-1100.

[6] Das. S, *Functional Fractional Calculus for System Identification and Controls*, Springer, Berlin 2008.

[7] Farina L. and Rinaldi S., *Positive Linear Systems; Theory and Applications*, J. Wiley: New York, 2000.

[8] Gantmacher F. R., *Theory of Matrices*, vol. I and II, Chelsea: New York, 1960.

[9] Kaczorek T., *Vectors and Matrices in Automatics and*

Electrotechnics, WNT, Warszawa 1998 (in Polish).
[10] Kaczorek T., *Positive 1D and 2D Systems*, Springer, London 2002.
[11] Kaczorek T., "Reachability and controllability to zero of positive fractional discrete-time systems", *Machine Intelligence and Robotics Control*, vol. 6, no.4, 2008, pp. 139-143.
[12] Kaczorek T., "Fractional positive continuous-time linear systems and their reachability", *Int. J. Appl. Math. Comput. Sci.*, vol. 18, no. 2, 2008, pp. 223-228.
[13] Kaczorek T., "Reachability and controllability to zero tests for standard and positive fractional discrete-time systems", *Journal of Automation and System Engineering*, vol. 42, no. 6-7-8, 2008, pp. 769-787.
[14] Olbrot A., "On degeneracy and related problems for linear constant time-lag systems". *Ricerche di Automatica*, vol. 3, no. 3, 1972, pp. 203-220.
[15] Podlubny I., *Fractional Differential Equations*, Academic Press, San Diego 1999.
[16] Popov V. M., "Pointwise degeneracy of linear time-invariant delay-differential equations", *Journal of Diff. Equations*, issue 11, 1972, pp. 541-561.
[17] Sabatier J., Agrawal O. P. and Machado J. A. T. (Eds), *Advances in Fractional Calculus, Theoretical Developments and Applications in Physics and Engineering*, Springer London 2007.
[18] Weiss L., "Controllability for various linear and nonlinear systems models", *Lecture Notes in Mathematics*, vol. 144, *Seminar on Differential Equations and Dynamic System II*, Springer, Berlin 1970, pp. 250-262.