Reachability and Controllability of Positive Fractional Discrete-Time Systems with Delay

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Abstract:

In the paper the positive fractional discrete-time linear systems with delay described by the state equations are considered. The solution to the state equations is derived using the Z transform. Necessary and sufficient conditions are established for the positivity, reachability and controllability to zero for fractional systems with one delay in state. The considerations are illustrated by an example.

Keywords: *fractional, positive discrete-time, linear system, reachability, controllability to zero.*

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values for non-negative initial conditions and non-negative controls. Examples of positive systems are given in monograph [3] and quoted there literature.

Positive linear systems are defined on cones and not on linear spaces. Therefore, theory of positive systems is more complicated and less advanced. Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays have been considered in [1, 3, 9,11].

The three definitions generally used for the fractional integro-differential operators are the Grünvald definition, the Riemann-Liouville definition and the Caputo definition. The mathematical fundamentals of fractional calculus are given in the monographs [7, 8, 12]. This idea has been used by engineers for modelling different process and designing fractional order controllers for time-delay systems [5, 10]. In the papers [4, 6] a new class of fractional positive systems described by the state equations were introduced and necessary and sufficient conditions for reachability and controllability were given.

In this paper using recent results, given in [1, 3, 4, 9], a problem of reachability and controllability to zero of fractional positive discrete-time systems with one delay in state will be considered. The paper is organized as follows: in section 2 using the definition of the fractional discrete derivative and Z transform the solution to state equations of the fractional systems is derived. The necessary and sufficient conditions for the positivity, reachability and controllability to zero of the fractional systems are established in sections 3, 4 and 5, respectively. A numerical example is given in section 6.

2. Preliminaries

Let $\Re^{N\times m}$ be the set of $N\times m$ matrices with entries from the field of real numbers and $\Re^N=\Re^{N\times 1}$. The set of $N\times m$ real matrices with non-negative entries will be denoted by $\Re_+^{N\times m}$ and $\Re_+^{N\times m} = \Re_+^{N\times 1}$. The set of non-negative integers will be denoted by Z_+ and the $N\times N$ identity matrix by I.

In this paper using the Grünwald-Letnikov approach the following definition of the fractional discrete derivative will be used [7, 8, 12]

$$\Delta^n x_i = \frac{1}{h^n}\sum_{j=0}^{i}(-1)^j\binom{n}{j}x_{i-j} = \frac{1}{h^n}\sum_{j=0}^{i}\omega_j^{(n)}x_{i-j}, \tag{1}$$

where $n\in R$ is the order of the fractional difference, h is the sampling interval and $i\in Z_+$ is the number of samples for which the derivative is calculated.

According to this definition, it is possible to obtain a discrete equivalent of the derivative (when n is positive), a discrete equivalent of the integration (when n is negative) and, when $n=0$, the original function.

The binomial coefficients $\omega_j^{(n)}$ can be obtained from [7]

$$\omega_j^{(n)} = \begin{cases} 1 & \text{for } j=0 \\ (-1)^j\dfrac{n(n-1)\cdots(n-j+1)}{j!} & \text{for } j=1,2,\ldots \end{cases} \tag{2}$$

or using the following recursive formula [12]

$$\omega_0^{(n)} = 1,\ \omega_j^{(n)} = (1-\frac{1+n}{j})\omega_{j-1}^{(n)},\ \ j=1,2,\ldots \tag{3}$$

For simplicity it will be assumed that $h=1$.

Consider the fractional discrete-time linear system with delay in state, described by the state-space equations

$$\Delta^n x_{i+1} = A_0x_i + A_1x_{i-1} + Bu_i,\ \ i\in Z_+, \tag{4a}$$

$$y_i = Cx_i + Du_i, \tag{4b}$$

where $x_i\in\Re^N$, $u_i\in\Re^m$, $y_i\in\Re^p$, $A_0\in\Re^{N\times N}$, $A_1\in\Re^{N\times N}$, $B\in\Re^{N\times m}$, $C\in\Re^{p\times N}$, $D\in\Re^{p\times m}$, with the initial conditions

$$x_{-i} = x(-i)\in\Re^N,\ \ i=0,1. \tag{5}$$

Using the definition (1) for $h=1$ we may write the equations (4) in the form

$$x_{i+1} + \sum_{j=1}^{i+1}(-1)^j\binom{n}{j}x_{i-j+1} = A_0x_i + A_1x_{i-1} + Bu_i,\ \ i\in Z_+, \tag{6a}$$

$$y_i = Cx_i + Du_i. \tag{6b}$$

Computing coefficients $\omega_j^{(n)}$ for $j=1,2$ and putting in order of delays in state, equation (6a) can be rewritten as follows

$$\begin{aligned} x_{i+1} &= (A_0 - \omega_1^{(n)} I)x_i + (A_1 - \omega_2^{(n)} I)x_{i-1} - \\ &- \sum_{j=3}^{i+1} \omega_j^{(n)} x_{i-j+1} + Bu_i = (A_0 + \omega_1 I)x_i + \\ &+ (A_1 + \omega_2 I)x_{i-1} + \sum_{j=3}^{i+1} \omega_j x_{i-j+1} + Bu_i \\ &= (A_0 + nI)x_i + (A_1 - \binom{n}{2} I)x_{i-1} + \\ &+ \sum_{j=3}^{i+1} \omega_j x_{i-j+1} + Bu_i, \quad i \in Z_+, \end{aligned} \tag{7}$$

where $\omega_j = -\omega_j^{(n)}$ and $\omega_j^{(n)}$ are given by (2) or (3).

Note that the fractional discrete-time linear system (6) is the classical discrete-time system with delays increasing with the number of samples $i \in Z_+$ [1]. The state equation in this case has the form

$$x_{i+1} = \sum_{h=0}^{i} \overline{A}_h x_{i-h}, \quad i \in Z_+, \tag{8}$$

where

$$\overline{A}_0 = A_0 + nI, \quad \overline{A}_1 = A_1 - \binom{n}{2} I, \quad \overline{A}_h = \omega_{h+1} I, \tag{9}$$
$$h = 2,3,\ldots,i.$$

From (2) it follows that coefficients $\omega_j = -\omega_j^{(n)}$ strongly decrease to zero when j increases to infinity.

Theorem 1. The solution of equation (6a) with initial conditions (5) is given by

$$x_i = \Phi_i x_0 + \Phi_{i-1}(A_1 + \omega_2 I)x_{-1} + \sum_{j=0}^{i-1} \Phi_{i-1-j} B u_j, \tag{10}$$

where the fundamental (transition) matrix Φ_i is determined by the equation

$$\begin{aligned} \Phi_{i+1} &= (A_0 + nI)\Phi_i + (A_1 + \omega_2 I)\Phi_{i-1} + \sum_{j=3}^{i+1} \omega_j \Phi_{i-j+1} = \\ &= \Phi_i (A_0 + nI) + \Phi_{i-1}(A_1 + \omega_2 I) + \sum_{j=3}^{i+1} \omega_j \Phi_{i-j+1}, \end{aligned} \tag{11}$$

with the initial conditions

$$\Phi_0 = I, \quad \Phi_i = 0 \text{ for } i < 0, \tag{12}$$

where $\omega_j = -\omega_j^{(n)}$ and $\omega_j^{(n)}$ are given by (2) or (3).

Proof. In the same way as in [2, 4] the proof will be accomplished using the Z transform.

Let $X(z)$ be the Z transform of x_i defined by

$$X(z) = Z[x_i] = \sum_{i=0}^{\infty} x_k z^{-i}. \tag{13}$$

Using the Z transform to (6a) and taking into account nonzero initial conditions (5) we obtain

$$\begin{aligned} &zX(z) - zx_0 + \sum_{j=1}^{i+1} \omega_j^{(n)} z^{-(j-1)} X(z) + \omega_2^{(n)} x_{-1} = \\ &= A_0 X(z) + A_1 z^{-1}(X(z) + x_{-1} z) + BU(z). \end{aligned} \tag{14}$$

Solving the equation (14) for $X(z)$ we obtain

$$\begin{aligned} X(z) &= \left[I\Delta^n(z^{-1}) - A_0 z^{-1} + A_1 z^{-2} \right]^{-1} \cdot \\ &\cdot \left(x_0 + (A_1 + \omega_2 I) z^{-1} x_{-1} + Bz^{-1} U(z) \right), \end{aligned} \tag{15}$$

where

$$\Delta^n(z^{-1}) = \sum_{j=0}^{i+1} \omega_j^{(n)} z^{-j}. \tag{16}$$

Let

$$\left[I\Delta^n(z^{-1}) - A_0 z^{-1} + A_1 z^{-2} \right]^{-1} = \sum_{i=0}^{\infty} \Phi(i) z^{-i}. \tag{17}$$

Substituting of (17) to (15) yields

$$X(z) = \sum_{i=0}^{\infty} \Phi(i) z^{-i} \left(x_0 + (A_1 + \omega_2 I) z^{-1} x_{-1} + Bz^{-1} U(z) \right). \tag{18}$$

Using the inverse Z transform to (18) we obtain the desired formula (10).

Substituting of (17) to the equality

$$[I\Delta^n(z^{-1}) - A_0 z^{-1} + A_1 z^{-2}][I\Delta^n(z^{-1}) - A_0 z^{-1} + A_1 z^{-2}]^{-1} = I \tag{19}$$

we obtain

$$\left[I\left(\sum_{j=0}^{i+1} \omega_j^{(n)} z^{-j} \right) - A_0 z^{-1} + A_1 z^{-2} \right] \left(\sum_{j=0}^{\infty} \Phi_i z^{-i} \right) = I. \tag{20}$$

Comparing the coefficients at the same powers of z^{-i} for $i=0,1,\ldots$, of the equality (20) we obtain

$$\begin{aligned} &\Phi_0 = I, \quad \Phi_1 = A_0 + In, \\ &\Phi_2 = (A_0 + In)\Phi_1 + (A_1 + \omega_2 I)\Phi_0, \end{aligned} \tag{21}$$

and in general case the equation (11).

The proof for second part of (11) is similar as given in [2] in the case of classical discrete-time system with delays.

Note that the solution (10) of fractional state equations can be derived using the recursive formula (7) for x_i, $i=0,1,2\ldots$ and the initial conditions (5) without applying the inverse Z transform.

3. Positivity of the fractional systems

Definition 1. [3, 4] The fractional system (4) is called the (internally) positive fractional system if and only if $x_i \in \Re_+^N$ and $y_i \in \Re_+^p$, $i \in Z_+$ for any initial conditions $x_{-i} \in \Re_+^N$, $i=0,1,$ and all input sequences $u_i \in \Re_+^m$, $i \in Z_+$.

The following two lemmas will be used in the proof of the positivity of the fractional system (4) with delay in state.

Lemma 1. [4] If the order of the fractional difference n satisfies

$$0 < n \leq 1 \tag{22}$$

then coefficients ω_j are positive, i.e.

$$\omega_j = -\omega_j^{(n)} = (-1)^{j+1}\binom{n}{j} > 0, \quad j = 1,2,\ldots \tag{23}$$

The proof of the lemma is given in [4].

Lemma 2. If the order of the fractional difference n satisfies the condition (22) and

$$A_0 + nI \in \Re_+^{N\times N}, \quad A_1 + \omega_2 I \in \Re_+^{N\times N} \tag{24}$$

then fundamental matrices have only nonnegative entries, i.e.

$$\Phi_i \in \Re_+^{N\times N}, \quad i \in Z_+. \tag{25}$$

The proof of the lemma follows from (11) and it is given in [4] using the mathematical induction.

Theorem 2. The fractional discrete-time system (4) for $0 < n \leq 1$ is positive if and only if

$$\begin{gathered} A_0 + In \in \Re_+^{N\times N}, \quad A_1 + \omega_2 I \in \Re_+^{N\times N}, \\ B \in \Re_+^{N\times m}, \quad C \in \Re_+^{p\times N}, \quad D \in \Re_+^{p\times m}. \end{gathered} \tag{26}$$

Proof. Sufficiency: If the condition (24) is satisfied then by Lemma 2 $\Phi_i \in \Re_+^{N\times N}$ holds for $i=0,1,2,\ldots$ If (25) and (26) are satisfied, then from (10) and (6b) we have $x_i \in \Re_+^N$ and $y_i \in \Re_+^p$ for every $i \in Z_+$ since $x_0 \in \Re_+^N$, $x_{-1} \in \Re_+^N$ by Definition 1 and $u_i \in \Re_+^m$, $i \in Z_+$.

Necessity: Let $u_i=0$ for $i \in Z_+$. Assuming that the system is positive from (7) for $i=0$ we obtain $x_1=(A_0+nI)x_0+(A_1+\omega_2 I)x_{-1}$ and from (6b) we have $y_0=Cx_0 \in \Re_+^p$. This implies $A_0+In \in \Re_+^{N\times N}$, $A_1+\omega_2 I \in \Re_+^{N\times N}$ and $C \in \Re_+^{p\times N}$ since $x_0 \in \Re_+^N$ and $x_{-1} \in \Re_+^N$ by definition 1 are arbitrary. Assuming $x_0, x_{-1}=0$ from (7) for $i=0$ we obtain $x_1=Bu_0 \in \Re_+^N$ and from (6b) we have $y_0=Du_0 \in \Re_+^p$, which implies $B \in \Re_+^{N\times m}$ and $D \in \Re_+^{p\times m}$, since $u_0 \in \Re_+^m$ by Definition 1 is arbitrary. ■

4. Reachability of the positive fractional systems

Let e_i, $i=1,2,\ldots,N$, be the ith column of the identity matrix I. A column ae_i for $a>0$ is called the monomial column.

Taking into account papers [1, 4, 11] we may formulate the following definition of reachability of the positive fractional system with delay.

Definition 2. The fractional system (4) is called reachable if for every state $x_f \in \Re_+^N$ there exists a natural number q and an input sequence $u_i \in \Re_+^m$, $i=0,1,2,\ldots,q-1$, which steers the state of the system (4) from zero initial states (5) (i.e. $x_{-1}=x_0=0$) to the desired final state $x_f \in \Re_+^N$.

Theorem 3. The positive fractional system (4) for $0 < n \leq 1$ is reachable in q steps if and only if the reachability matrix

$$R_q := [B, \Phi_1 B, \ldots, \Phi_{q-1} B] \tag{27}$$

contains N linearly independent monomial columns.

Proof. The solution of equation (6a) has the form (10). For zero initial conditions $x_{-1}=x_0=0$ and $i=q$ we have

$$x_f = x_q = \sum_{j=0}^{q-1} \Phi_{q-j-1} B u_j = R_q u_0^q, \tag{28}$$

where the reachability matrix has the form (27) and an input sequence has the following form

$$u_0^q = \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix}. \tag{29}$$

From Definition 2 and (28) it follows that for every $x_f \in \Re_+^N$ there exists an input sequence $u_i \in \Re_+^m$, $i=0,1,2,\ldots,q-1$, if and only if the matrix R_q (27) contains N linearly independent monomial columns. ■

If the fractional system (4) is reachable and $R_q^T [R_q R_q^T]^{-1} \in \Re_+^{qm\times N}$ then the nonnegative input u_0^q (29), which steers the state of the system (4) from zero initial states (5) (i.e. $x_{-1}=x_0=0$) to the desired final state $x_f \in \Re_+^N$, is given by the formula [1]

$$u_0^q = R_q^T [R_q R_q^T]^{-1} x_f. \tag{30}$$

5. Controllability to zero of the positive fractional systems

Taking into account papers [4, 9, 11] we may formulate the following definition of controllability to zero of the positive fractional system with delay.

Definition 3. The fractional system (4) is called controllable to zero in $q>0$ steps if for any nonzero initial states $x_{-1}, x_0 \in \Re_+^N$ there exists an input sequence $u_i \in \Re_+^m$, $i=0,1,\ldots,q-1$, which steers the state of the system from nonzero initial conditions (5) to zero ($x_f=0$).

Theorem 4. The positive fractional system (4) for $0 < n \leq 1$ is controllable to zero in $q>0$ steps if and only if

$$\Phi_q = 0, \quad \Phi_{q-1}(A_1 + \omega_2 I) = 0. \tag{31}$$

Moreover $u_i=0$ for $i=0,1,\ldots,q-1$.

Proof. From equation (10) for $x_f = 0$ and $i = q$ we have

$$0 = \Phi_q x_0 + \Phi_{q-1}(A_1 + \omega_2 I)x_{-1} + R_q u_0^q, \tag{32}$$

where the matrix R_q has the form (27) and u_0^q is defined by (29).

It is well known that for finite q and $A_1+\omega_2 I\in\Re_+^{N\times N}$, $\Phi_i\in\Re_+^{N\times N}$, $x_{-1},x_0\in\Re_+^N$, $R_q\in\Re_+^{N\times qm}$ do not exist positive $u_0^q\in\Re_+^{qm}$ satisfying equation (32).

The equation (32) is satisfied for any nonzero initial conditions (5) and $R_q\in\Re_+^{N\times qm}$ if and only if the conditions (31) hold and $u_0^q\in 0$.

Lemma 3. The positive fractional system with delay in state (4) for $0 < n \le 1$ is controllable to zero:

a) in $q=1$ step if and only if

$$A_0 + nI = 0,\ \ A_1 + \omega_2 I = 0, \tag{33}$$

b) in $q=2$ steps if and only if

$$(A_0 + nI)^2 = 0,\ \ A_1 + \omega_2 I = 0, \tag{34}$$

c) in an infinite number of steps if and only if the system is asymptotically stable.

Proof. Using (11) for $i = 1,2,\ldots$ we obtain fundamental matrices Φ_i of the forms:

$$\begin{aligned}
&\Phi_1 = A_0 + nI,\\
&\Phi_2 = \Phi_1^2 + (A_1 + \omega_2 I)\Phi_0,\\
&\Phi_3 = \Phi_1^3 + \Phi_1(A_1 + \omega_2 I) + (A_1 + \omega_2 I)\Phi_1 + \omega_3\Phi_0,\\
&\Phi_4 = \Phi_1^4 + \Phi_1^2(A_1 + \omega_2 I) + \Phi_1(A_1 + \omega_2 I)\Phi_1 +\\
&+ (A_1 + \omega_2 I)\Phi_1^2 + (A_1 + \omega_2 I)^2 + 2\omega_3\Phi_1 + \omega_4\Phi_0,\\
&\qquad\vdots\\
&\Phi_q = \Phi_1^q + \Phi_1^{q-2}(A_1 + \omega_2 I) + \ldots + \omega_q\Phi_0.
\end{aligned} \tag{35}$$

From the above it follows that the conditions (31) can be satisfied if and only if the conditions (33) hold and $q=1$.

In case b) the matrix A_0+nI is a nilpotent matrix with the index of nilpotence $\mu=2$ and we have $\Phi_2=\Phi_2^1+(A_1+\omega_2 I)\Phi_0=0$. Hence the conditions (31) will be satisfied if and only if the conditions (34) hold and $q=2$.

In case c) if the system is asymptotically stable then

$$\lim_{q\to\infty}\Phi_q x_0 = 0 \ \text{ and } \ \lim_{q\to\infty}\Phi_{q-1}(A_1 + \omega_2 I)x_{-1} = 0 \tag{36}$$

for every $x_{-1},x_0\in\Re_+^N$. Moreover $\Phi_q\to 0$ for $q\to\infty$ and $\omega_q\to 0$. Hence equation (32) is satisfied for $u_0^q = 0$ and by Theorem 4 the system is controllable in an infinite number of steps. ■

6. Example

Test reachability and controllability to zero of positive fractional system with delay in state (4) with the matrices

$$A_0 = \begin{bmatrix} -1/2 & 3/10 \\ 0 & -1/2 \end{bmatrix},\ A_1 = \begin{bmatrix} -1/8 & 0 \\ 0 & -1/8 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{37}$$

First, we check for which values of the order of the fractional difference n this system is positive.

From Theorem 2 we have

$$\begin{aligned}
&A_0 + In = \begin{bmatrix} -1/2+n & 3/10 \\ 0 & -1/2+n \end{bmatrix} \in \Re_+^{2\times 2},\\
&A_1 + \omega_2 I = \begin{bmatrix} -1/8+\omega_2 & 0 \\ 0 & -1/8+\omega_2 \end{bmatrix} \in \Re_+^{2\times 2}
\end{aligned} \tag{38}$$

if and only if $n=1/2$, since $\omega_2 = -\omega_2^{(n)} = \dfrac{(1-n)n}{2} \ge 1/8$.

Using (27) for $q=2$ we obtain the reachability matrix

$$R_2 = [B, \Phi_1 B] = \begin{bmatrix} 0 & 3/10 \\ 1 & 0 \end{bmatrix} \tag{39}$$

which contains two linearly independent monomial columns. Therefore, by Theorem 3 the positive fractional system (37) is reachable in two steps for $n=1/2$.

Computing u_0^q from (30) for the final state $x_f = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ we obtain

$$u_0^2 = R_2^{-1}x_f = \begin{bmatrix} 0 & 3/10 \\ 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 10/3 \end{bmatrix}. \tag{40}$$

We check out received results. Using (7) for matrices (37) with $n=1/2$, $x_{-1}=x_0=0$ and the input sequence $u_0=10/3$ and $u_1=2$ we obtain

$$x_1 = Bu_0 = \begin{bmatrix} 0 \\ 10/3 \end{bmatrix},\ \ x_2 = (A_0 + nI)x_1 + Bu_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{41}$$

Next, we test the controllability to zero of this system for $n=1/2$.

Using (11) for $i=1,2,\ldots$ we obtain fundamental matrices Φ_i of the forms:

$$\begin{aligned}
&\Phi_1 = A_0 + nI = \begin{bmatrix} 0 & 3/10 \\ 0 & 0 \end{bmatrix},\\
&\Phi_2 = \Phi_1^2 + (A_1 + \omega_2 I)\Phi_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\\
&\Phi_3 = \omega_3\Phi_0 = \begin{bmatrix} 1/16 & 0 \\ 0 & 1/16 \end{bmatrix},\\
&\Phi_4 = 2\omega_3\Phi_1 + \omega_4\Phi_0 = \begin{bmatrix} 5/128 & 3/80 \\ 0 & 5/128 \end{bmatrix}\\
&\qquad\vdots
\end{aligned} \tag{42}$$

From the above it follows that the case b) of Lemma 3 is satisfied. Therefore, the positive fractional system for $n=1/2$ with the matrices (37) is controllable to zero in 2 steps.

To verify obtained result we find the solution of equation (4a) for $n=1/2$ with matrices (37) and $x_{-1}=\begin{bmatrix}2\\3\end{bmatrix}$, $x_0=\begin{bmatrix}3\\1\end{bmatrix}$, $u_0=0$ and $u_1=0$.

Using (7) for $i=0,1$ we obtain, respectively

$$x_1=(A_0+nI)x_0+(A_1+\omega_2 I)x_{-1}=$$

$$=\begin{bmatrix}0 & 3/10\\0 & 0\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}+\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}3/10\\0\end{bmatrix},$$

$$x_2=(A_0+nI)x_1+(A_1+\omega_2 I)x_0=$$

$$=\begin{bmatrix}0 & 3/10\\0 & 0\end{bmatrix}\begin{bmatrix}3/10\\0\end{bmatrix}+\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

7. Concluding remarks

The concept of positive fractional system has been extended for the discrete-time linear systems with delays. Necessary and sufficient conditions for the positivity (Theorem 2), reachability (Theorem 3) and controllability to zero (Theorem 4) for order of the fractional difference n satisfied the following condition $0 < n \le 1$, have been established.

A formula for computing a nonnegative input u_0^q (30), which steers the state of the system (4) from zero initial states (5) (i.e. $x_{-1}=x_0=0$) to the desired final state $x_f=\Re_+^N$ has been given.

The considerations can be easily extended for the positive fractional systems with multiple delays in state and control.

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