Positive Discrete-time Linear Lyapunov Systems

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Abstract:

The concept of a positive discrete-time Lyapunov system is introduced. Solution of the Lyapunov state equation is derived and necessary and sufficient conditions for the positivity of Lyapunov system are established. Different necessary and sufficient conditions for the asymptotic stability of the positive Lyapunov systems are given. Using the Kronecker product of matrices necessary and sufficient conditions for reachability and controllability of the positive Lyapunov systems are established. The considerations are illustrated by numerical example.

Keywords: *Lyapunov, positive, realization, existence.*

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [3, 4]. The realization problem for positive linear systems without and with time delays has been considered in [1, 4, 7-10].

The reachability and minimum energy control of positive linear discrete-time systems was considered in [2].

The controllability and observability of Lyapunov systems have been investigated in the paper [11].

In this paper the notion of positive discrete-time Lyapunov system will be introduced and necessary and sufficient conditions for the positivity, asymptotic stability and reachability and controllability of positive Lyapunov systems will be established.

To the best knowledge of the author those problems for the positive Lyapunov systems has not been considered yet.

2. Discrete-time linear Lyapunov systems

Let $R^{n\times m}$ be the set of $n\times m$ real matrices and $R^n := R^{n\times 1}$. The set of $n\times m$ real matrices with nonnegative entries will be denoted by $R_+^{n\times m}$ and $R_+^n := R_+^{n\times 1}$. The set of nonnegative integers will be denoted by Z_+ and the $n\times n$ identity matrix will be denoted by I_n.

Consider the discrete-time linear Lyapunov system described by the equations

$$X_{i+1}=A_0X_i+X_iA_1+BU_i \tag{1a}$$

$$Y_i=CX_i+DU_i,\ i\in Z_+ \tag{1b}$$

where $X_i\in R^{n\times n}$, $U_i\in R^{m\times n}$, $Y_i\in R^{p\times n}$ are the state, input and output matrices and $A_k\in R^{n\times n}$, $k=0,1$; $B\in R^{n\times m}$, $C\in R^{p\times n}$, $D\in R^{p\times m}$.

Theorem 1. Solution of the equation (1a) satisfying the initial condition X_0 is given by the formula

$$X_i=\sum_{k=0}^{i}\frac{i!}{k!(i-k)!}A_0^kX_0A_1^{i-k}+\sum_{j=0}^{i-1}\sum_{k=0}^{j}\frac{j!}{k!(j-k)!}A_0^kBU_{i-j-1}A_1^{j-k}\quad ,i\in Z_+ \tag{2}$$

Proof. The proof will be accomplished by induction. The hypothesis is true for $i=1,2$, since from (2) we obtain

$$X_1=A_0X_0+X_0A_1+BU_0,\ X_2=A_0^2X_0+2A_0X_0A_1+ +X_0A_1^2+A_0BU_0+BU_0A_1+BU_1$$

The same result we obtain from (1a) for $i=0,1$. Assuming that the hypothesis is true for $i=n>1$ we shall show that it is also valid for $i=n+1$. From assumption we have

$$X_n=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}A_0^kX_0A_1^{n-k}+\sum_{j=0}^{n-1}\sum_{k=0}^{j}\frac{j!}{k!(j-k)!}A_0^kBU_{n-j-1}A_1^{j-k} \tag{3}$$

Using (1a) and (3) we obtain

$$X_{n+1}=A_0X_n+X_nA_1+BU_n=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}A_0^{k+1}X_0A_1^{n-k}+\sum_{j=0}^{n-1}\sum_{k=0}^{j}\frac{j!}{k!(j-k)!}A_0^{k+1}BU_{n-j-1}A_1^{j-k}+\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}A_0^kX_0A_1^{n-k+1}+\sum_{j=0}^{n-1}\sum_{k=0}^{j}\frac{j!}{k!(j-k)!}A_0^kBU_{n-j-1}A_1^{j-k+1}+BU_n=\sum_{k=0}^{n+1}\frac{(n+1)!}{k!(n-k+1)!}A_0^kX_0A_1^{n-k+1}+\sum_{j=0}^{n}\sum_{k=0}^{j-1}\frac{j!}{k!(j-k)!}A_0^kBU_{n-j}A_1^{j-k}$$

This complies the proof. ■

Substituting of (2) into (1b) yields the output formula

$$Y_i = \sum_{k=0}^{i} \frac{i!}{k!(i-k)!} C A_0^k X_0 A_1^{i-k} + \sum_{j=0}^{i-1} \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} C A_0^k B U_{i-j-1} A_1^{j-k} + D U_i \tag{4}$$

3. Positive discrete-time linear Lyapunov systems

Definition 1. The Lyapunov system (1) is called (internally) positive if for any $X_0 \in R_+^{n\times n}$ and all inputs $U_i \in R_+^{m\times n}$, $i \in Z_+$ we have $X_i \in R_+^{n\times n}$ and $Y_i \in R_+^{p\times n}$ for $i \in Z_+$.

The Kronecker product $A \otimes B$ of the matrices $A=[a_{ij}] \in R^{m\times n}$ and $B \in R^{p\times q}$ is the block matrix [5]

$$A \otimes B = [a_{ij} B]_{\substack{i=1,\dots,m \\ j-1,\dots,n}} \in R^{mp\times nq} \tag{5}$$

Lemma 1. Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be the eigenvalues of a matrix $A \in R^{m\times m}$ and $\mu_1, \mu_2, \dots, \mu_n$ be the eigenvalues of a matrix $B \in R^{n\times n}$. Then $\lambda_i + \mu_j$ for $i=1,\dots,m\dots$ and $j=1,\dots,n$ are the eigenvalues of the matrix

$$A \otimes I_n + I_m \otimes B^T \tag{6}$$

Using the Kronecker product we shall prove the following theorem.

Theorem 2. The Lyapunov system (1) is positive if and only if

$$A_k \in R_+^{n\times n},\ k=1,0;\ B \in R_+^{n\times m},\ C \in R_+^{p\times n},\ D \in R_+^{p\times m}$$

Proof. Using the Kronecker product (5) we may write the equations (1) in the equivalent form [5]

$$\begin{aligned} x_{i+1} &= \overline{A} x_i + \overline{B} u_i \\ y_i &= \overline{C} x_i + \overline{D} u_i \end{aligned}, \ i \in Z_+ \tag{7}$$

where

$$\overline{A} = A_0 \otimes I_n + I_n \otimes A_1^T \in R^{\overline{n}\times\overline{n}},\quad \overline{B} = B \otimes I_n \in R^{\overline{n}\times\overline{m}},$$

$$\overline{C} = C \otimes I_n \in R^{\overline{p}\times\overline{n}},\quad \overline{D} = D \otimes I_n \in R^{\overline{p}\times\overline{m}}$$

$$x_i = [X_{i1}\ X_{i2}\ \dots\ X_{in}]^T,\quad u_i = [U_{i1}\ U_{i2}\ \dots\ U_{im}]^T, \tag{8}$$

$$y_i = [Y_{i1}\ Y_{i2}\ \dots\ Y_{ip}]^T$$

$$\overline{n} = n^2,\quad \overline{m} = nm,\quad \overline{p} = np$$

and X_{ij}, U_{ij} and Y_{ij} are the jth rows of the matrices X_i, U_i and Y_i respectively and T is denotes the transpose.

It is well known [4] that the system (7) is positive if and only if

$$\overline{A} \in R_+^{\overline{n}\times\overline{n}},\quad \overline{B} \in R_+^{\overline{n}\times\overline{m}},\quad \overline{C} \in R_+^{\overline{p}\times\overline{n}},\quad \overline{D} \in R_+^{\overline{p}\times\overline{m}} \tag{9}$$

From (8) it follows that the conditions (7) are equivalent to the conditions (9). ■

Theorem 3. Let $A = A_0 + A_1$ and

$$p_{\overline{A}}(z) = \det\left[I_{\overline{n}} z - \overline{A}\right] = z^{\overline{n}} + a_{\overline{n}-1} z^{\overline{n}-1} + \dots + a_1 z + a_0 \tag{10}$$

be the characteristic polynomial of the positive system (1) (the matrix $\overline{A} = A_0 \otimes I_n + I_n \otimes A_1^T$).

If

$$I_n \otimes A_1^T = A_1 \otimes I_n \tag{11}$$

then

$$p_{\overline{A}}(z) = (p_A(z))^n,\ (p_A(z) = \det\left[I_n z - A\right]) \tag{12}$$

and

$$p_{\overline{A}}^{(k)}(A) = 0 \quad \text{for } k=0,1,\dots,n \tag{13}$$

where

$$p_{\overline{A}}^{(k)}(z) = \frac{d^k p_{\overline{A}}(z)}{dz^k} \quad \text{for } k=0,1,\dots,n\text{-}1 \tag{14}$$

Proof. If (11) holds then

$$\overline{A} = A_0 \otimes I_n + I_n \otimes A_1^T = (A_0 + A_1) \otimes I_n = A \otimes I_n$$

and

$$p_{\overline{A}}(z) = \det\left[I_{\overline{n}} z - \overline{A}\right] = \det\left[(I_n z - A) \otimes I_n\right] = \left(\det\left[I_n z - A\right]\right)^n = (p_A(z))^n$$

since $\det[A \otimes B] = \det[A]^n \det[B]^n$ for $A, B \in R^{n\times n}$ [5].

Using the well-known equality [5] $(A \otimes B)(C \otimes D) = AC \otimes BD$ it easy to show that

$$\overline{A}^k = (A \otimes I_n)^k = A^k \otimes I_n \quad \text{for } k=0,1,\dots,\overline{n} \tag{15}$$

From Cayley-Hamilton theorem we have

$$p_{\overline{A}}(\overline{A}) = \overline{A}^{\overline{n}} + a_{\overline{n}-1} \overline{A}^{\overline{n}-1} + \dots + a_1 \overline{A} + a_0 I_{\overline{n}} = 0 \tag{16}$$

Substituting (15) into (16) we obtain

$$p_{\overline{A}}(\overline{A}) = \overline{A}^{\overline{n}} \otimes I_n + a_{\overline{n}-1}(\overline{A}^{\overline{n}-1} \otimes I_n) + \dots + a_1 \overline{A} \otimes I_n + a_0 I_{\overline{n}} \otimes I_n = p_{\overline{A}}(A) \otimes I_n = 0 \tag{17}$$

From (17) and (12) we have $p_{\overline{A}}(A) = (p_A(A))^n = 0$ and this implies (13). ■

Theorem 3 for $k=0$ is an extension of classic Cayley-Hamilton theorem for positive Lyapunov systems.

Remark 1. It is easy to show that condition (11) is met if and only if matrix A_1 is a scalar matrix, i.e. $A_1 = aI_n$, $a \neq 0$.

Remark 2. From (12) it follows that the spectrum σ_A of the matrix A is a subset of the spectrum $\sigma_{\overline{A}}$ of the matrix $\overline{A}$, i.e. $\sigma_A \subset \sigma_{\overline{A}}$.

Example 1. For the matrices

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

the condition (11) is satisfied since

$$I_n \otimes A_1^T = A_1 \otimes I_n = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

In this case the characteristic polynomials $p_A(z)$ and $p_{\overline{A}}(z)$ are

$$p_A(z) = \det[I_n z - (A_0 + A_1)] = \begin{vmatrix} z-2 & -1 \\ 1 & z \end{vmatrix}$$

$$= (z-1)^2 = z^2 - 2z + 1$$

$$p_{\overline{A}}(z) = \det[I_{\overline{n}} z - \overline{A}] = \begin{vmatrix} z-2 & 0 & -1 & 0 \\ 0 & z-2 & 0 & -1 \\ 1 & 0 & z & 0 \\ 0 & 1 & 0 & z \end{vmatrix}$$

$$= (z-1)^4 = (p_A(z))^2 = z^4 - 4z^3 + 6z^2 - 4z + 1$$

Using (13) and taking into account that

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix}$$

we obtain

$$p_{\overline{A}}(A) = A^4 - 4A^3 + 6A^2 - 4A + I_2$$

$$= \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix} - 4\begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix} + 6\begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$$

$$-4\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$p_{\overline{A}}^{(1)}(A) = 4A^3 - 12A^2 + 12A - 4I_n$$

$$= 4\begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix} - 12\begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} + 12\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$-4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In this particular case we have also

$$p_{\overline{A}}^{(2)}(A) = 12A^2 - 24A + 12I_n = 12\begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$$

$$-24\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} + 12\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

since $p_{\overline{A}}^{(2)}(z) = 12(z\text{-}1)^2 = 12p_A(z)$

Example 2. For the matrices

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

the condition (11) is not satisfied since

$$I_n \otimes A_1^T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \text{ and } A_1 \otimes I_n = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The characteristic polynomial (10) has the form

$$p_{\overline{A}}(z) = \begin{vmatrix} z-2 & 0 & -1 & 0 \\ 0 & z-3 & 0 & -1 \\ 1 & 0 & z & 0 \\ 0 & 1 & 0 & z-1 \end{vmatrix}$$

$$= z^4 - 6z^3 + 13z^2 - 12z + 4$$

Taking into account that

$$A = A_0 + A_1 = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 3 & 6 \\ -6 & -3 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 0 & 9 \\ -9 & -9 \end{bmatrix}$$

and using (13) for $k=0$ we obtain

$$p_{\overline{A}}(A) = A^4 - 6A^3 + 13A^2 - 12A + 4I_2$$

$$= \begin{bmatrix} 0 & 9 \\ -9 & -9 \end{bmatrix} - 6\begin{bmatrix} 3 & 6 \\ -6 & -3 \end{bmatrix} + 13\begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix} - 12\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$+4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, the equation (13) for $k=0$ is not satisfied.

4. Asymptotic stability

Consider the positive autonomous Lyapunov system

$$X_{i+1} = A_0 X_i + X_i A_1 \ , i \in Z_+ \tag{18}$$

where $X_i \in R_+^{n \times n}$, $A_k \in R_+^{n \times n}$, $k=1,0$.

Definition 2. The positive Lyapunov system (18) is called asymptotically stable if

$$\lim_{i \to \infty} (A_0 \otimes I_n + I_n \otimes A_1^T)^i x_0 = 0 \quad \text{for all } x_0 \in R_+^{\overline{n}} \tag{19}$$

where x_0 is defined by (8).

Theorem 4. The Lyapunov system (18) is asymptotically stable if and only if

$$|z_{0i} + z_{1j}| < 1 \quad \text{for } i,j=1,\ldots,n \tag{20}$$

where z_{0i}, $i=1,\ldots,n$ (z_{1j}, $j=1,\ldots,n$) are the eigenvalues of the matrix A_0 (A_1).

Proof. By Lemma 1 $z_{0i}+z_{1j}$ $(ij=1,...,n)$ are the eigenvalues of the matrix $\overline{A}=A_0\otimes I_n+I_n\otimes A_1^T$. The condition (19) is satisfied if and only if the condition (20) is met. Thus by Definition 2 the positive Lyapunov system (18) is asymptotically stable if and only if the condition (20) is satisfied. ■

Applying the well-known theorem [4, Theorem 2.13] for the equivalent positive system $x_{i+1}=\overline{A}x_i$ we obtain the following.

Theorem 5. The positive Lyapunov system (18) is asymptotically stable if and only if all coefficients $\overline{a}_i$ $(i=0,1,...\overline{n}-1)$ of the characteristic polynomial

$$\overline{p}(z)=\det\left[I_{\overline{n}}z-\overline{A}\right]=z^{\overline{n}}+\overline{a}_{\overline{n}-1}z^{\overline{n}-1}+...+\overline{a}_1z+\overline{a}_0 \quad (21)$$

are positive.

Let

$$\hat{A}=I_{\overline{n}}-A_0\otimes I_n+I_n\otimes A_1^T=[\hat{a}_{ij}]_{\substack{i=1,...,\overline{n}\\ j=1,...,\overline{n}}} \quad (22)$$

Then using the well-known Theorem [4, p.69, Theorem 2.14] we obtain the following.

Theorem 6. The positive Lyapunov system (18) is asymptotically stable if and only if all principal minors of the matrix (22) are positive i.e.

$$|\hat{a}_{11}|>0,\ \begin{vmatrix}\hat{a}_{11} & \hat{a}_{12}\\ \hat{a}_{21} & \hat{a}_{22}\end{vmatrix}>0,\ \begin{vmatrix}\hat{a}_{11} & \hat{a}_{12} & \hat{a}_{13}\\ \hat{a}_{21} & \hat{a}_{22} & \hat{a}_{23}\\ \hat{a}_{13} & \hat{a}_{23} & \hat{a}_{33}\end{vmatrix}>0,...,\det\hat{A}>0 \quad (23)$$

From Theorem 2.15 [4, p.70] we have following theorem that gives sufficient condition for instability of the positive Lyapunov system (18).

Theorem 7. The positive Lyapunov system (18) is unstable if at least on diagonal entry of the matrix $\overline{A}=A_0\otimes I_n+I_n\otimes A_1^T$ is greater 1 i.e. $\overline{a}_{kk}>1$ for some $k\in(1,2,...,\overline{n})$.

Example 3. Consider the positive system (18) with

$$A_0=\begin{bmatrix}0.1 & 1\\ 0 & 0.2\end{bmatrix},\quad A_1=\begin{bmatrix}0.3 & 0\\ 2 & 0.4\end{bmatrix} \quad (24)$$

In this case $z_{01}=0.1$, $z_{02}=0.2$ and $z_{11}=0.3$, $z_{12}=0.4$ and the condition (20) is satisfied since $z_{01}+z_{11}=0.4$, $z_{01}+z_{12}=0.5$, $z_{02}+z_{11}=0.5$, $z_{02}+z_{12}=0.6$. Therefore by Theorem 4 the system (18) with (24) is asymptotically stable.

The characteristic polynomial (21) for (24) has the form (25)

$$\overline{p}(z)=\det\left[I_{\overline{n}}z-\overline{A}\right]=\begin{vmatrix}z+0.6 & -2 & -1 & 0\\ 0 & z+0.5 & 0 & -1\\ 0 & 0 & z+0.5 & -2\\ 0 & 0 & 0 & z+0.4\end{vmatrix}$$
$$=(z+0.4)(z+0.5)^2(z+0.6)$$
$$=z^4+2z^3+1.49z^2+0.49z+0.06$$

The coefficients of the polynomial (25) are positive and by Theorem 5 the positive system (18) with (24) is asymptotically stable.

The matrix (22) for (24) has the form

$$\hat{A}=I_{\overline{n}}-A_0\otimes I_n+I_n\otimes A_1^T=\begin{bmatrix}0.6 & -2 & -1 & 0\\ 0 & 0.5 & 0 & -1\\ 0 & 0 & 0.5 & -2\\ 0 & 0 & 0 & 0.4\end{bmatrix}$$

and its principal minors are equal to

$$|\hat{a}_{11}|=0.60,\ \begin{vmatrix}\hat{a}_{11} & \hat{a}_{12}\\ \hat{a}_{21} & \hat{a}_{22}\end{vmatrix}=0.30,\ \begin{vmatrix}\hat{a}_{11} & \hat{a}_{12} & \hat{a}_{13}\\ \hat{a}_{21} & \hat{a}_{22} & \hat{a}_{23}\\ \hat{a}_{13} & \hat{a}_{23} & \hat{a}_{33}\end{vmatrix}=0.15,$$

$$\det\hat{A}=0.06$$

Therefore, by Theorem 6 the positive system (18) with (24) is asymptotically stable.

Example 4. Consider the positive system (18) with

$$A_0=\begin{bmatrix}0.4 & 1\\ 0 & 0.6\end{bmatrix},\quad A_1=\begin{bmatrix}0.5 & 0\\ 2 & 0.6\end{bmatrix} \quad (26)$$

The matrix $\overline{A}$ for (26) has the form

$$\overline{A}=A_0\otimes I_n+I_n\otimes A_1^T=\begin{bmatrix}0.9 & 2 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1.1 & 2\\ 0 & 0 & 0 & 1.2\end{bmatrix} \quad (27)$$

The matrix (27) has two diagonal entries $\overline{a}_{33}=1.1$, $\overline{a}_{44}=1.2$ greater 1. Thus by Theorem 7 the positive system (18) with (26) is unstable. The same result we obtain using the Theorems 4, 5 and 6.

5. Reachability

Consider the positive Lyapunov system (1).

Definition 3. The positive Lyapunov system (1) is called reachable if for any given $X_f\in R_+^{n\times n}$ there exists $q\in Z_+$, $q>0$ and an input sequence $U_i\in R_+^{n\times m}$, $i=0,1,...,q$-1 that steers the state of the system from $X_0\neq 0$ to X_f, i.e. $X_q=X_f$.

Let e_i, $i=1,...,n$ be the i^{th} column of the identity matrix I_n. The column $ae_i\in R_+^n$ is called monomial.

Theorem 8. The positive Lyapunov system (1) is reachable if and only if the reachability matrix

$$R_{\overline{n}}=\left[\overline{B}\quad \overline{A}\overline{B}\quad ...\quad \overline{A}^{\overline{n}-1}\overline{B}\right]\ (\overline{n}=n^2) \quad (28)$$

contains $\overline{n}$ linearly independent monomial columns, where $\overline{B}$ and $\overline{A}$ are defined by (8).

Proof. From Definition 3 if follows that the system (1) is reachable if and only if the equivalent system (7) is reachable. It is well known [4] that the system (7) is reachable if and only if the matrix (28) contains $\overline{n}$ linearly independent monomial columns. ■

Remark 3. Other different well-known [4] tests of the reachability of the positive system (7) can also be applied to the positive Lyapunov system (1).
Let $\bar{u} \in R_+^{\bar{n}\times n}$ be the matrix consisting from $\bar{n}$ rows of the matrix

$$u_0^{\bar{n}} = \begin{bmatrix} u_{\bar{n}-1} \\ \vdots \\ u_0 \end{bmatrix} \qquad (29)$$

corresponding to the $\bar{n}$ chosen linearly independent monomial column of the matrix (28). Then from the equation $R_{\bar{n}} u_0^{\bar{n}} = x_f$ we have

$$\bar{R}_{\bar{n}} \bar{u} = x_f \qquad (30)$$

where $\bar{R}_{\bar{n}} \bar{u} = x_f$ is the matrix consisting form linearly independent monomial column of (28) and

$$x_f = \begin{bmatrix} X_{f1} & X_{f2} & \dots & X_{fn} \end{bmatrix}^T, \ (X_{fi} \text{ is the ith row of } X_f) \qquad (31)$$

From (30) we obtain

$$\bar{u} = \bar{R}_{\bar{n}}^{-1} x_f \qquad (32)$$

where $\bar{R}_{\bar{n}}^{-1} \in R_+^{\bar{n}\times\bar{n}}$ is a monomial matrix.

Using (32) we can compute the desired input sequence that steers the state of the positive system (1) from $X_0=0$ to X_f.

Example 5. Consider the positive system (1) with

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (33)$$

In this case

$$\bar{A} = A_0 \otimes I_n + I_n \otimes A_1^T = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

$$\bar{B} = B \otimes I_n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using (28) we obtain the matrix

$$R_4 = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \bar{A}^2\bar{B} & \bar{A}^3\bar{B} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 3 & 0 & 9 & 0 & 27 & 0 \\ 0 & 0 & 0 & 4 & 0 & 16 & 0 & 64 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (34)$$

that contains four linearly independent monomial columns.

Therefore, by Theorem 8 the positive Lyapunov system (1) with (33) is reachable in two steps since the first four columns of (34) are linearly independent monomial columns.

For $X_f = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ from (32) we obtain

$$\bar{u} = \bar{R}_{\bar{n}}^{-1} x_f = \begin{bmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1/3 \\ 1/2 \end{bmatrix}$$

and the desired input sequence is

$$u_0 = \begin{bmatrix} 3 & 4 \end{bmatrix}, \quad u_1 = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \end{bmatrix}.$$

Theorem 9. The positive Lyapunov system (1) is unreachable if the matrix B has no monomial columns or the matrix $[A_0+A_1 \ B]$ has less than n linearly independent monomial columns.

Proof. Note that the matrix
$\bar{B} = B \otimes I_n \quad (\bar{A} = A_0 \otimes I_n + I_n \otimes A_1^T)$
has monomial columns if and only if matrix B (A_0+A_1) has monomial columns. In [6] it was shown that the positive system (7) is unreachable if the matrix $\bar{B}$ has no monomial columns or the matrix $[\bar{B} \ \ \bar{A}]$ has less then $\bar{n}$ linearly independent monomial columns. ■

Remark 4. Theorem 9 gives only necessary condition for reachability of the positive Lyapunov system (1). For example the system (1) with

$A_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ satisfies the conditions of Theorem 9 but it is unreachable.

Remark 5. The other different necessary and sufficient conditions for reachability of the positive system (7) given in [4 p.131, Theorem 3.3] can be extended for the positive Lyapunov system (1).

6. Controllability

Definition 4. The positive Lyapunov system (1) is called controllable if for any $X_0 \in R_+^{n\times n}$ and a final state $X_f \in R_+^{n\times n}$ there exists $q \in Z_+$, $q>0$ and a sequence of inputs $U_i \in R_+^{n\times m}$, $i=0,1,\dots,q-1$ that steers the state of the system from $X_0 \neq 0$ to X_f, i.e. $X_q = X_f$.

If $X_f=0$ then the system is called controllable to zero.

Lemma 2. The matrix $\bar{A}$ (defined by (8)) is nilpotent if and only if the matrices A_0 and A_1 are nilpotent.

Proof. If A_0 and A_1 are nilpotent then by definition their eigenvalues $z_{0i}=0$ and $z_{1j}=0$ for $i,j=1,\dots,n$. By Lemma 1 the eigenvalues of the matrix $\bar{A}$ are also zero and it is nilpotent matrix.

Let the matrix $\overline{A}$ be nilpotent and $z_{0i}+z_{1j}=0$ for $i,j=1,...,n$ but at list one of the matrix is not nilpotent. If, for example, the matrix A_0 is not nilpotent then $z_{0i} \neq 0$ for $i=1,...,n$ and from Lemma 1 it follows that $z_{0i}=-z_{1j} \neq 0$ for some $i,j \in [1,...,n]$. This contradicts the assumption that system (1) is positive and $A_0,\ A_1 \in R_+^{n\times n}$. ■

Theorem 10. The positive Lyapunov system (1) is controllable in $\overline{n}$ steps if and only if the matrices A_0 and A_1 are nilpotent and reachability matrix (28) contains $\overline{n}$ linearly independent monomial columns.

Proof. Using the equivalent system (7) for the positive Lyapunov system (1) we obtain

$$x_f = \overline{A}^{\overline{n}} x_0 + R_{\overline{n}} u_0^{\overline{n}} \tag{35}$$

where $R_{\overline{n}}$ and $u_0^{\overline{n}}$ are defined by (28) and (29), respectively. By Lemma 2 $\overline{A}^{\overline{n}}=0$ if and only if the matrices A_0 and A_1 are nilpotent. From (35) it follows that there exists an input sequence $\overline{n}$ (consisting from $\overline{n}$ rows of $u_0^{\overline{n}}$ corresponding to the $\overline{n}$ chosen linearly independent monomial columns of $R_{\overline{n}}$) if and only if the matrix (28) contains $\overline{n}$ linearly independent monomial columns. ■

Remark 5. If the matrices A_0 and A_1 are not nilpotent but the positive Lyapunov system (1) is asymptotically stable and the reachability matrix (28) contains $\overline{n}$ linearly independent monomial columns then the system is controllable in an infinite number of steps.

From comparison of Theorem 8 and 10 we have the following corollary

Corollary. The controllability of the positive Lyapunov system (1) implies its reachability.

Example 6. Consider the positive Lyapunov system (1) with

$$A_0 = \begin{bmatrix} 0 & a_{01} \\ a_{02} & a_{03} \end{bmatrix},\quad A_1 = \begin{bmatrix} 0 & a_{11} \\ a_{12} & a_{13} \end{bmatrix},\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{36}$$

For which values of entries a_{0k}, a_{1k}, $k=1,2,3$ the system is controllable in $\overline{n}=2n=4$ steps.

In this case

$$\overline{A} = A_0 \otimes I_n + I_n \otimes A_1^T = \begin{bmatrix} 0 & a_{12} & a_{01} & 0 \\ a_{11} & a_{13} & 0 & a_{01} \\ a_{02} & 0 & a_{03} & a_{12} \\ 0 & a_{02} & a_{11} & a_{03}+a_{13} \end{bmatrix},$$

$$\overline{B} = B \otimes I_n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix $\overline{A}$ is nilpotent if $a_{11}=a_{13}=a_{02}=a_{03}=0$ and $a_{01}\geq 0, a_{12}\geq 0$.

The reachability matrix (28) has the form

$$R_{\overline{n}} = \begin{bmatrix} \overline{B} & \overline{A}\overline{B} & \overline{A}^2\overline{B} & \overline{A}^3\overline{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & a_{01} & 0 & 0 & 2a_{01}a_{12} & 0 & 0 \\ 0 & 0 & 0 & a_{01} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & a_{12} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{37}$$

and it contains four linearly independent monomial columns for $a_{12}=0$. Hence by Theorem 10 the positive system (1) with (36) is controllable in four steps if $a_{11}=a_{12}=a_{13}=a_{02}=a_{03}=0$ and $a_{01}\geq 0$.

The considerations can be extended for the dual notion of observability of the positive Lyapunov system (1).

7. Concluding remarks

The notion of a positive discrete-time Lyapunov system has been introduced. The solution (2) of the Lyapunov state equation (1a) has been derived. Necessary and sufficient conditions for the positivity of the system (1) (Theorem 2) and for the asymptotic stability (Theorem 4) have been established. Using Kronecker product of matrices and the concept of equivalent positive system necessary and sufficient conditions for the reachability and observability have been formulated and proved. The considerations have been illustrated by numerical examples.

An extension of those results for positive continuous-time Lyapunov systems will be considered in the next paper. An extension of those considerations for 2D positive Lyapunov systems is also possible.

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