Simple Conditions for Robust Stability of Linear Positive Discrete-Time Systems with One Delay

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Abstract:

Simple new necessary and sufficient conditions for robust stability of the positive linear discrete-time systems with one delay in the general case and in the two special cases: 1) linear unity rank uncertainty structure, 2) linear uncertainty structure with non-negative perturbation matrices, are established. The conditions are based on the new simple criterion for asymptotic stability of the positive linear discrete-time systems with one delay, proved in the paper. The considerations are illustrated by numerical examples.

Keywords: *robust stability, linear system, positive, discrete-time, delay*

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values for non-negative initial states and non-negative controls.

The conditions for asymptotic stability and robust stability of positive discrete-time systems with delays were given in [1-14, 17].

The main purpose of the paper is to give the quite new simple necessary and sufficient conditions for robust stability of linear positive discrete-time systems with one delay with linear uncertainty structure in the general case and in two special cases: 1) unity rank uncertainty structure, 2) non-negative perturbation matrices.

In the paper the following notations will be used: $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with non-negative entries and $\Re_+^n=\Re_+^{n\times 1}$; Z_+ - the set of non-negative integers. A matrix $A=[a_{ij}]\in\Re^{n\times m}$ with $a_{ij}>0$ for all $i=1,2,...,n$; $j=1,2,...,m$, will be called strictly positive and denoted by $A>0$. Similarly, a vector $x\in\Re^n$ with positive (negative) all components will be called strictly positive (negative) and denoted by $x>0$ ($x<0$).

2. Problem formulation

Consider an uncertain positive discrete-time linear system with one delay $h\geq 1$ (h is a positive integer) described by the homogeneous equation

$$x_{i+1}=A_0(q)x_i+A_h(q)x_{i-h},\quad q\in Q,\quad i\in Z_+ \tag{1}$$

where $x_i\in\Re^n$ is the state vector, $q=[q_1,q_2,...,q_m]$ is the vector of uncertain parameters,

$$Q=\{q\colon q_r\in[q_r^-,q_r^+],\, q_r^-<q_r^+,\, r=1,2,\ldots,m\} \tag{2}$$

is the value set of uncertain parameters and $A_k(q)\in\Re_+^{n\times n}$, $(k=0,h)$ for any fixed $q\in Q$.

The initial conditions for (1) have the form $x_{-k}\in\Re^n$ for $k=h,h-1,...,0$.

If $A_0(q)\in\Re_+^{n\times n}$ and $A_h(q)\in\Re_+^{n\times n}$ for all $q\in Q$ then the solution of equation (1) satisfies the condition $x_i\in\Re_+^n$, $i\in Z_+$, for any non-negative initial conditions, i.e.

$$x_{-k}\in\Re_+^n,\, k=h,h-1,...,0. \tag{3}$$

The positive system (1) is called robustly stable if the condition $\lim_{i\to\infty}x_i=0$ holds for all initial conditions (3) and for any fixed $q\in Q$.

It is well known that the positive system (1) is robustly stable if and only if for any fixed $q\in Q$ all zeros $z_k(q)$ $(k=1,2,...,\tilde{n}=(h+1)n)$ of the characteristic equation

$$w(z,q)=\det[z^{h+1}I_n-A_0(q)z^h-A_h(q)]=0 \tag{4}$$

satisfy the condition $|z_k(q)|<1, k=1,2,...,\tilde{n}=(h+1)n$.

The positive system without delay equivalent to (1) has the form

$$\tilde{x}_{i+1}\; A(q)\tilde{x}_i\;,\; q\in Q,\; i\in Z_+, \tag{5}$$

where the state vector $\tilde{x}_i\in\Re_+^{\tilde{n}}$ with $\tilde{n}=(h+1)n$ and

$$A(q)=\begin{bmatrix} A_0(q) & 0 & \cdots & 0 & A_h(q)\\ I_n & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix},\quad \tilde{x}_i=\begin{bmatrix} x_i\\ x_{i-1}\\ \vdots\\ x_{i-h+1}\\ x_{i-h}\end{bmatrix}. \tag{6}$$

The positive system (5) is robustly stable if and only if

$$w_A(z,q)=\det(zI_{\tilde{n}}-A(q))\neq 0,\ \text{for } |z|\geq 1,\ \forall q\in Q \tag{7}$$

It is easy to see that $w(z,q)=w_A(z,q)$ (see for example [17] for the system without uncertain parameters). Hence, robust stability of the positive system (1) (with delays) is equivalent to robust stability of the positive system (5) (without delays).

From the above and by generalisation of known condition for asymptotic stability of positive systems without delays [15, 16] to the positive system (5) with uncertain parameters we obtain the following theorem.

Theorem 1. The positive system with delays (1) is robustly stable if and only if the following equivalent conditions hold:

1) all leading principal minors $\Delta_i(q)$ $(i=1,2,...,\tilde{n}= =(h+1)n)$ of the matrix

$$\bar{A}(q) = I_{\tilde{n}} - A(q) = \begin{bmatrix} I_n - A_0(q) & 0 & \cdots & 0 & -A_h(q) \\ -I_n & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \\ 0 & 0 & \cdots & -I_n & I_n \end{bmatrix} \tag{8}$$

are positive for all $q \in Q$,

2) all coefficients of the characteristic polynomial of the matrix $S(q)=-\bar{A}(q)$, of the form

$$\tilde{w}_S(z,q) = \det[zI_{\tilde{n}} - S(q)] = \det[(z+1)I_{\tilde{n}} - A(q)] = = z^{\tilde{n}} + \sum_{k=0}^{\tilde{n}-1} \tilde{a}_k(q) z^k, \tag{9}$$

are positive for all $q \in Q$, i.e. $\tilde{a}_k(q) > 0$, $\forall q \in Q$, $k=0,1,..., \tilde{n}-1$.

The aim of the paper is to give the new necessary and sufficient conditions for robust stability of the positive discrete-time system (1) with delays in the general case (with any uncertainty structure in matrices $A_0(q)$ and $A_h(q)$) and very simple conditions for robust stability of the system with linear uncertainty structure, i.e. with

$$A_0(q) = A_{00} + \sum_{r=1}^{m} q_r E_{0r}, \quad A_h(q) = A_{h0} + \sum_{r=1}^{m} q_r E_{hr}, \tag{10}$$

where $A_{k0} \in \Re_+^{n \times n}$ and $E_{kr} \in \Re^{n \times n}$ $(k=0,h,\ r=1,2,...,m)$ are the nominal and the perturbation matrices, respectively, in two cases:

1) unity rank uncertainty structure, i.e.

$$\text{rank}\, E_{kr} = 1 \ \text{ for } \ k=0,h,\ r=1,2,...,m, \tag{11}$$

2) non-negative perturbation matrices, i.e. $E_{kr} \in \Re_+^{n \times n}$ for $k=0,h,\ r=1,2,...,m$ (satisfaction of (11) is not necessary).

3. Solution of the problem

Let us consider the positive discrete-time linear system with delay $h \geq 1$

$$x_{i+1} = A_0 x_i + A_h x_{i-h},\ i \in Z_+, \tag{12}$$

where $A_0,\ A_h \in \Re_+^{n \times n}$.

The main result of the paper is based on the following theorem.

Theorem 2. The positive discrete-time system (12) with delay is asymptotically stable if and only if the positive discrete-time system without delay

$$x_{i+1} = (A_0 + A_h) x_i,\ i \in Z_+, \tag{13}$$

is asymptotically stable.

Proof. In [14] (see also [13]) it was shown that the positive discrete-time system with delay (12) is asymptotically stable if and only if there a strictly positive vector $\lambda > 0$ $(\lambda \in \Re_+^n)$ exists, such that $(A_0 + A_h - I)\lambda < 0$.

In [18] it was shown that the positive discrete-time system without delay $x_{i+1} = Ax_i$ is asymptotically stable if and only if there $\lambda \in \Re_+^n$, $\lambda > 0$ exists, such that $(A - I)\lambda < 0$.

Hence, the conditions above are equivalent for $A = A_0 + A_h$. This means that asymptotic stability of the positive discrete-time system (12) with delay is equivalent to asymptotic stability of the positive discrete-time system (13) without delay. ■

The proof of Theorem 2 is very simple in comparison with the proof given in [3].

Generalisation of Theorem 2 to the case of positive discrete-time systems with multiple delays is given in [7].

From Theorem 2 it follows that asymptotic stability of the positive system (12) with delay does not depend of the size of time delay $h \geq 1$. Such a kind of stability is called as asymptotic stability independent of delay.

Moreover, asymptotic stability of the positive system (13) without delay is equivalent to asymptotic stability not only of the positive system (12) but also of the positive systems: $x_{i+1} = A_h x_i + A_0 x_{i-h}$; $x_{i+1} = 0.5 A_h x_i + +(A_0 + 0.5 A_h) x_{i-h}$ or $x_{i+1} = (A_h + A_0) x_{i-h}$, for example.

3.1. Robust stability in the general case

By generalisation of Theorem 2 to the positive system (1) with uncertain parameters we obtain the following theorem.

Theorem 3. The positive discrete-time system (1) with delay is robustly stable if and only if the positive discrete-time system without delay, described by the state equation

$$x_{i+1} = D(q) x_i,\ q \in Q,\ i \in Z_+, \tag{14}$$

is robustly stable, where

$$D(q) = A_0(q) + A_h(q). \tag{15}$$

From (10) and (15) it follows that $D(q) = D_0 + D_h(q)$ where

$$D_0 = A_{00} + A_{h0},\ D_h(q) = \sum_{r=1}^{m} q_r E_r, \quad E_r = E_{0r} + E_{hr}. \tag{16}$$

The positive system (14) is robustly stable if and only if all roots $z_k(q)$ $(k=1,2,...,n)$ of the characteristic polynomial

$$w(z,q) = \det[zI_n - D(q)] \tag{17}$$

satisfy the condition $|z_k(q)| < 1$, $k=1,2,...,n$ for any fixed $q \in Q$.

By generalisation of known conditions for asymptotic stability of positive systems without delays [15, 16] to the positive system (14) with uncertain parameters and using Theorem 3 we obtain the following theorem.

Theorem 4. The positive system (1) is robustly stable if and only the following equivalent conditions hold:

1) all leading principal minors $\Delta_i(q)$ $(i=1,2,...,n)$ of the matrix

$$\bar{D}(q)=I_n-D(q)=I_n-(A_0(q)+A_h(q)) \tag{18}$$

are positive for all $q\in Q$,

2) all coefficients of the characteristic polynomial of the matrix $S(q)=-\bar{D}(q)$, of the form

$$w_S(z,q)=\det[zI_n-S(q)]=\det[(z+1)I_n-D(q)]= \\ =z^n+\sum_{k=0}^{n-1}a_k(q)z^k, \tag{19}$$

are positive for all $q\in Q$, i.e. $a_k(q)>0$, $\forall q\in Q, k=0,1,...,$ $n-1$.

Proof. The conditions 1) and 2) are necessary and sufficient for robust stability of the positive system (14). These conditions were obtained by generalisation of known conditions for asymptotic stability of the positive discrete-time systems without delays, given in [15, 16]. The proof it follows from the above and Theorem 3. ■

Lemma 1. If the condition $d_{kk}(q)\le 1$ for all $q\in Q$ $(k=1,2,..., n)$ where $d_{kk}(q)$ are diagonal entries of the matrix $D(q)$ does not hold then the positive system (1) is not robustly stable.

Proof. The proof it follows from Theorem 3 and generalisation of simple necessary condition for asymptotic stability of the positive discrete-time systems without delays, given in [15, 16]. ■

Note that checking of robust stability of the positive system (1) on the basis of Theorem 4 is very simple in comparison with application of Theorem 1. This follows from the fact that the size of the system (14) is extremely less than the size of the system (5) (equal to $\tilde{n}=(h+1)n$).

The conditions 1) and 2) of Theorem 4 can be rewritten in the form

$$\min_{q\in Q}\Delta_i(q)>0,\ \ i=1,2,\dots,n, \\ \min_{q\in Q}a_k(q)>0,\ \ k=0,1,\dots,n-1. \tag{20}$$

For checking the conditions (20) we can apply the computer methods for finding the minimal values with constraints of multivariable functions.

3.2. Robust stability in the case of linear unity rank uncertainty structure

Let us consider the positive discrete-time system (1) with matrices of the form (10).

In the case of linear unity rank uncertainty structure the matrix (15) can be written in the form

$$D(q)=D_0+\sum_{r=1}^{m}q_rE_r,\ \ q\in Q, \tag{21}$$

where $D_0=A_{00}+A_{h0}$ is asymptotically stable nominal matrix, $q=[q_1,q_2,...,q_m]$ is the vector of deviations of uncertain parameters from nominal values and known perturbation matrices $E_r=E_{0r}+E_{hr}$ $(r=1,2,...,m)$ satisfy the condition

$$\operatorname{rank}E_r=1 \ \ \text{for}\ \ r=1,2,...,m. \tag{22}$$

The set (2) is m-dimensional hyperrectangle with $L=2^m$ vertices. Let us denote by $\bar{q}_l=[q_1^l,q_2^l,\dots,q_m^l]$, with $q_r^l=q_r^-$ or $q_r^l=q_r^+$ for $r=1,2,...,m$, $l=1,2,...,L$, the vertices of hyperrectangle (2). Moreover, by $D_l=D(\bar{q}_l)$, $l=1,2,...,L$, denote the non-negative vertex matrices, corresponding to the vertices of the set (2).

Theorem 5. The positive system (1) with linear unity rank uncertainty structure is robustly stable if and only if all the vertex matrices $D_l=D(\bar{q}_l)$, $l=1,2,...,L$, are asymptotically stable.

Proof. *Necessity.* Necessity is obvious because the vertex matrices belong to the family $\{D(q): q\in Q\}$ of the state space matrices of the positive system (14).

Sufficiency. If the matrix (21) has linear unity rank uncertainty structure then the coefficients $a_k(q)$, $k=0,1,..., n-1$ of characteristic polynomial (19) are real multilinear functions of uncertain parameters q_r, $r=1,2,$ $...,m$, and

$$\min_{q\in Q}a_k(q)=\min_{l}a_k(\bar{q}_l),\ \ k=0,1,\dots,n-1.$$

From the above and condition 2) of Theorem 4 it follows that if all the vertex matrices $D_l=D(\bar{q}_l)$, $l=1,2,$ $...,L$, are asymptotically stable then the positive system (1) with linear unity rank uncertainty structure is robustly stable. ■

To asymptotic stability checking of the vertex matrices we can apply Theorem 4 putting $D_l=D(\bar{q}_l)$, $l=1,2,...,L$, instead of $D(q)$.

From Theorem 5 it follows that robust stability of the positive system (1) with linear unity rank uncertainty structure is equivalent to robust stability of $L=2^m$ positive discrete-time systems

$$x_{i+1}=D_lx_i,\ \ i\in Z_+,\ \ l=1,2,...,L=2^m, \tag{23}$$

where $D_l=D(\bar{q}_l)$ is the vertex matrix corresponding to the l-th vertex $\bar{q}_l$ of the value set (2) of uncertain parameters.

3.3. Robust stability in the case of linear uncertainty structure with non-negative perturbation matrices

In the case of system (1) with linear uncertainty structure with non-negative perturbation matrices the matrix $D(q)$ has the form (15), (16) with $E_r\in\Re_+^{n\times n}$ for $r=1,2,...,m$. Satisfaction of (11) is not necessary.

In such a case $q_r^- E_r \le q_r E_r \le q_r^+ E_r$ for any fixed $q_r \in [q_r^-, q_r^+]$. Therefore, $D(q) \in D_I$ for any fixed $q \in Q$ where $D_I = [D^-, D^+]$ is the non-negative interval matrix with

$$D^- = D_0 + \sum_{r=1}^{m} q_r^- E_r, \tag{24}$$

$$D^+ = D_0 + \sum_{r=1}^{m} q_r^+ E_r. \tag{25}$$

Recall that $D(q) \in D_I$ if and only if $d_{ij}^- \le d_{ij}(q) \le d_{ij}^+$, where d_{ij}^-, d_{ij}^+ and $d_{ij}(q)$ are then entries of the matrices D^-, D^+ and $D(q)$ respectively.

Note that the interval matrix $D_I = [D^-, D^+]$ is non-negative if and only the matrix (24) is non-negative.

Robust stability of non-negative interval matrix $D_I = [D^-, D^+]$ is equivalent to asymptotic stability of the matrix D^+(see [8, 10], for example). Therefore, we have the following theorem.

Theorem 6. If the perturbation matrices E_r $(r = 1,2,...,m)$ are non-negative, then the positive system (1) with the matrices (10) is robustly stable if and only if the matrix (25) is asymptotically stable, where D_0 and E_r are defined in (16).

4. Illustrative examples

Example 1. Check robust stability of the positive system (1) with the matrices

$$A_0 = \begin{bmatrix} 0.1 + q_1 q_2 & 0.2 + q_2 \\ 0.2 + q_1^2 & 0.1 + q_1 \end{bmatrix}, \qquad A_h = \begin{bmatrix} 0.4 + q_2 & 0 \\ 0 & 0.5 + q_1 q_2 \end{bmatrix}, \tag{26}$$

with $h \ge 1$, $q = [q_1, q_2] \in Q$, where

$$Q = \{q : q_1 \in [-0.1, 0.1], q_2 \in [-0.1, 0.1]\}. \tag{27}$$

The matrix (18) for the system has the form

$$\overline{D}(q) = I_n - (A_0(q) + A_h(q)) = \begin{bmatrix} 0.5 - q_1 q_2 - q_2 & -0.2 - q_2 \\ -0.2 - q_1^2 & 0.4 - q_1 - q_1 q_2 \end{bmatrix}. \tag{28}$$

Computing the leading principal minors of the matrix (28) we obtain

$$\Delta_1(q) = 0.5 - q_1 q_2 - q_2,$$

$$\Delta_2(q) = \det \overline{D}(q) = 0.16 - 0.5 q_1 - 0.6 q_2 + 0.1 q_1 q_2 - 0.2 q_1^2 + q_1 q_2^2 + q_1^2 q_2^2.$$

The minimal values of the above functions with $q \in Q$ are as follows:

$$\delta_1 = \min_{q \in Q} \Delta_1(q) = 0.39, \quad \delta_2 = \min_{q \in Q} \Delta_2(q) = 0.0501,$$

where minimal values are assigned for $q_1 = q_2 = 0.1$.

Hence, the leading principal minors of the matrix (28) are positive and the system is robustly stable, according to Theorem 4.

Example 2. Check robust stability of the positive system (1) with the matrices

$$A_0 = \begin{bmatrix} 0.1 + q_2 & 0.2 + q_2 \\ 0.2 + q_1 & 0.1 + q_1 \end{bmatrix}, \qquad A_h = \begin{bmatrix} 0.4 + q_2 & 0 \\ 0 & 0.5 + q_1 \end{bmatrix}, \tag{29}$$

with $h \ge 1$, $q = [q_1, q_2] \in Q$, where

$$Q = \{q : q_1 \in [-0.1, 0.1], q_2 \in [-0.1, 0.1]\}. \tag{30}$$

The matrix (15) for the system has the form

$$D(q) = A_0(q) + A_h(q) = \begin{bmatrix} 0.5 + 2q_2 & 0.2 + q_2 \\ 0.2 + q_1 & 0.6 + 2q_1 \end{bmatrix} = D_0 + q_1 E_1 + q_2 E_2, \tag{31}$$

where

$$D_0 = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.6 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}. \tag{32}$$

The nominal matrix D_0 is asymptotically stable (all eigenvalues have absolute values less than 1); the perturbation matrices have unity rank uncertainty structure with non-negative entries. To robust stability checking we can apply Theorem 5 or Theorem 6.

First, we apply Theorem 5. The set (30) has the following vertices: $\overline{q}_1 = [-0.1, -0.1]$, $\overline{q}_2 = [-0.1, 0.1]$, $\overline{q}_3 = [0.1, 0.1]$, $\overline{q}_4 = [0.1, -0.1]$. The corresponding vertex matrices have the forms

$$D_1 = D(\overline{q}_1) = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.4 \end{bmatrix}, \quad D_2 = D(\overline{q}_2) = \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.4 \end{bmatrix},$$

$$D_3 = D(\overline{q}_3) = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.8 \end{bmatrix}, \quad D_4 = D(\overline{q}_4) = \begin{bmatrix} 0.3 & 0.1 \\ 0.3 & 0.8 \end{bmatrix}.$$

The matrix $D_3 = D(\overline{q}_3)$ has eigenvalues: 0.4459 and 1.0541. Because one eigenvalue is greater than 1, the matrix $D_3 = D(\overline{q}_3)$ is not asymptotically stable. From Theorem 5 we have that the system is not robustly stable.

Now we apply Theorem 6. Computing the matrix (25) we obtain

$$D^+ = D_0 + q_1^+ E_1 + q_2^+ E_2 = D_0 + 0.1(E_1 + E_2) = D_3.$$

The matrix $D^+ = D_3$ is not asymptotically stable, hence from Theorem 6 it follows that the system is not robustly stable.

5. Concluding remarks

Simple new necessary and sufficient conditions for robust stability of the positive discrete-time linear system (1) with linear uncertainty structure in the general case (Theorem 4) and in two special cases: 1) unity rank uncertainty structure (Theorem 5), 2) non-negative perturbation matrices (Theorem 6), have been given.

These conditions are based on the new simple criterion for asymptotic stability of the positive linear discrete-time systems with one delay (Theorem 2).

The proposed conditions are very simple in comparison to the existing conditions for robust stability (see Theorem 1 and paper [5], for example).

The considerations can be generalised to the positive discrete-time systems with multiple delays, using the results of the paper [7].

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