FRACTIONAL 2D LINEAR SYSTEMS

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Abstract:

A new class of 2D fractional linear systems is introduced. A notion of α order 2D difference is proposed. Fractional 2D state equations of linear system are given and their solutions are derived using 2D $\mathcal{Z}$ transform. The classical Cayley-Hamilton theorem is extended for the 2D fractional systems. Necessary and sufficient conditions for the reachability and controllability to zero of 2D fractional systems are established.

Keywords: *linear, 2D, fractional, system, fractional difference, reachability, controllability to zero*

1. Introduction

The most popular models of two-dimensional (2D) linear systems are the models introduced by Roesser [23], Fornasini-Marchesini [4, 5] and Kurek [22]. The models have been extended for positive systems in [10, 24, 13, 9]. An overview of 2D linear system theory is given in [1, 2, 7, 8] and some recent result in positive systems has been given in the monographs [3, 9] and in paper [24]. Reachability and minimum energy control of positive 2D systems with one delay in states have been considered in [13]. The notion of internally positive 2D system (model) with delay in states and in inputs has been introduced and necessary and sufficient conditions for the internal positivity, reachability, controllability, observability and the minimum energy control problem have been established in [13]. The notions of positive fractional discrete-time and continuous-time linear systems have been introduced in [14, 15]. The notion for 2D positive fractional linear systems has been extended in [11].

In this paper a new class of 2D fractional linear systems will be introduced. The paper is organized as follows. In section 2 the fractional 2D state equations are proposed and their solutions are derived. The classical Cayley-Hamilton theorem is extended for 2D fractional systems. In section 3 necessary and sufficient conditions for the reachability and controllability to zero of the 2D fractional systems are established. Concluding remarks are given in section 4.

2. Fractional 2D state equations and their solutions

Let $\Re^{n\times m}$ be the set of real $n \times m$ matrices and $\Re^{n} = \Re^{n\times 1}$. The set of nonnegative integers will be denoted by Z_+ and the $n \times m$ identity matrix will be denoted by I_n.

Definition 1. The α order 2D fractional difference of x_{ij} is defined by the formula

$$\Delta^{\alpha} x_{ij} = \sum_{k=0}^{i}\sum_{l=0}^{j} c_{\alpha}(k,l) x_{i-k,j-l}, \quad 0<\alpha\le 1 \tag{1a}$$

where

$$c_{\alpha}(k,l) = \begin{cases} 1 & \text{for } k=0 \text{ or/and } l=0 \\ (-1)^{k+l}\dfrac{\alpha(\alpha-1)\ldots(\alpha-k+1)\alpha(\alpha-1)\ldots(\alpha-l+1)}{k!l!} & \\ \text{for } k+l>0 & \end{cases} \tag{1b}$$

The justification of Definition 1 is given in Appendix A. In Appendix A it is also shown that if $0<\alpha<1$ then $c_{\alpha}(k,l)>0$ for $k+l>1$.

Consider the α order 2D fractional linear system, described by the state equations

$$\Delta^{\alpha} x_{i+1,j+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \tag{2a}$$

$$y_{ij} = C x_{ij} + D u_{ij} \tag{2b}$$

where $x_{ij}\in\Re^{n}$, $u_{ij}\in\Re^{m}$, $y_{ij}\in\Re^{p}$ are the state, input and output vectors and $A_k\in\Re^{n\times n}$, $B_k\in\Re^{n\times m}$, $k=0,1,2,$ $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Using Definition 1 we may write the equation (2a) in the form

$$\begin{aligned} x_{i+1,j+1} = \overline{A}_0 x_{ij} + \overline{A}_1 x_{i+1,j} + \overline{A}_2 x_{i,j+1} - \sum_{k=0}^{i+1}\sum_{\substack{l=0\\k+l>1}}^{j+1} c_{\alpha}(k,l) x_{i-k+1,j-l+1} + \\ + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \end{aligned} \tag{3}$$

where $\overline{A}_0 = A_0 - I_n\alpha^2$, $\overline{A}_k = A_k + I_n\alpha$, $k=1,2.$

From (1b) it follows that the coefficients (1b) in (1a) strongly decrease when k and l increase. Therefore, in practical problems it is assumed that i and j are bounded by some natural numbers L_1 and L_2. In this case (3) takes the form

$$\begin{aligned} x_{i+1,j+1} = \overline{A}_0 x_{ij} + \overline{A}_1 x_{i+1,j} + \overline{A}_2 x_{i,j+1} - \sum_{k=0}^{L_1+1}\sum_{l=0}^{L_2+1} c_{\alpha}(k,l) x_{i-k+1,j-l+1} + \\ + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \end{aligned} \tag{3a}$$

Note that the fractional systems are 2D linear systems with delays increasing with i and j.

The boundary conditions for the equation (3) and (3a) are given in the form

$$x_{i0},\ i\in Z_+ \quad \text{and} \quad x_{0j},\ j\in Z_+ \tag{4}$$

Theorem 1. The solution of equation (3) with boundary conditions (4) is given by

$$x_{ij} = \sum_{p=1}^{i} T_{i-p,j-1}(\overline{A}_1 x_{p0} + B_1 u_{p0}) + \sum_{q=1}^{j} T_{i-1,j-q}(\overline{A}_2 x_{0q} + B_2 u_{0q}) + \\ + \sum_{p=1}^{i-1} T_{i-p-1,j-1}\overline{A}_0 x_{p0} + \sum_{q=1}^{j-1} T_{i-1,j-q-1}\overline{A}_0 x_{0q} + T_{i-1,j-1}\overline{A}_0 u_{00} + \\ + \sum_{p=0}^{i-1}\sum_{q=0}^{j-1} T_{i-p-1,j-q-1} B_0 u_{pq} + \sum_{p=0}^{i}\sum_{q=0}^{j} (T_{i-p-1,j-q-1}B_1 + T_{i-p,j-q-1}B_2) u_{pq} \tag{5}$$

where the transition matrices T_{pq} are defined by the formula

$$T_{pq} = \begin{cases} I_n \text{ for } p = q = 0 \\ \overline{A}_0 T_{p-1,q-1} + \overline{A}_1 T_{p,q-1} + \overline{A}_2 T_{p-1,q} - \sum_{\substack{k=0 \\ k+l<p+q-2}}^{p-1}\sum_{l=0}^{q-1} c_\alpha(p-k, q-l) T_{kl} \\ \quad \text{for } p+q>0 \\ 0 \text{ (zero matrix) for } p<0 \text{ or/and } q<0 \end{cases} \tag{6}$$

and $\overline{A}_k = A_k - I_n\alpha$ for $k=0,1,2.$

Proof. Let $X(z_1, z_2)$ be the 2D $\mathcal{Z}$-transform of x_{ij} defined by

$$X(z_1,z_2) = \mathcal{Z}[x_{ij}] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{ij} z_1^{-i} z_2^{-j} \tag{7}$$

Taking into account that

$$\mathcal{Z}[x_{i+1,j+1}] = z_1 z_2 [X(z_1,z_2) - X(z_1,0) - X(0,z_2) + x_{00}] \\ \mathcal{Z}[x_{i+1,j}] = z_1[X(z_1,z_2) - X(0,z_2)], \quad X(0,z_2) = \sum_{j=0}^{\infty} x_{0j} z_2^{-j} \\ \mathcal{Z}[x_{i,j+1}] = z_2[X(z_1,z_2) - X(z_1,0)], \quad X(z_1,0) = \sum_{i=0}^{\infty} x_{i0} z_1^{-i} \\ \mathcal{Z}[x_{i-k,j-l}] = z_1^{-k} z_2^{-l} X(z_1,z_2) \tag{8}$$

then from (3) with (4) we obtain

$$X(z_1,z_2) = G^{-1}(z_1,z_2)\Big\{(B_0 + B_1 z_1 + B_2 z_2)U(z_1,z_2) - \\ - z_1[\overline{A}_1 B_1]\begin{bmatrix} X(0,z_2) \\ U(0,z_2)\end{bmatrix} - z_2[\overline{A}_2 B_2]\begin{bmatrix} X(z_1,0) \\ U(z_1,0)\end{bmatrix} + \\ + z_z z_2[X(z_1,0) + X(0,z_2) - x_{00}]\Big\} \tag{9}$$

where

$$G(z_1,z_2) = \\ = \left[I_n z_1 z_2 + \sum_{\substack{k=0 \\ k+l>1}}^{L_1+1}\sum_{l=0}^{L_2+1} I_n c_\alpha(k,l) z_1^{-(k-1)} z_2^{-(l-1)} - \overline{A}_0 - \overline{A}_1 z_1 - \overline{A}_2 z_2 \right], \tag{10}$$

and $U(z_1,z_2) = \mathcal{Z}[u_{ij}]$.

Let

$$G^{-1}(z_1,z_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)} \tag{11}$$

From the equality $G^{-1}(z_1,z_2)G(z_1,z_2) = G(z_1,z_2)G^{-1}(z_1,z_2) = I_n$ it follows that

$$\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)}\right) G(z_1,z_2) = \\ = G(z_1,z_2)\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)}\right) = I_n \tag{12}$$

Comparison of the coefficients at the same powers of z_1 and z_2 of (12) yields the formula (6).

Substituting (11) into (9) we obtain

$$X(z_1,z_2) = \\ = \left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)}\right)\Big\{(B_0 + B_1 z_1 + B_2 z_2)U(z_1,z_2) - \\ - z_1[\overline{A}_1 B_1]\begin{bmatrix} X(0,z_2) \\ U(0,z_2)\end{bmatrix} - z_2[\overline{A}_2 B_2]\begin{bmatrix} X(z_1,0) \\ U(z_1,0)\end{bmatrix} + z_1 z_2[X(z_1,0) + \\ + X(0,z_2) - x_{00}]\Big\} \tag{13}$$

Using the 2D inverse $\mathcal{Z}$ transform to (13) we obtain the desired formula (5). ■

From (10) we have

$$G(z_1,z_2) = z_1 z_2 \overline{G}(z_1,z_2) \tag{14}$$

where

$$\overline{G}(z_1,z_2) = I_n + \sum_{k=0}^{L_1+1}\sum_{l=0}^{L_2+1} I_n c_\alpha(k,l) z_1^{-k} z_2^{-l} - \\ - \overline{A}_0 z^{-1} z_2^{-1} - \overline{A}_1 z_2^{-1} - \overline{A}_2 z_1^{-1} \tag{15}$$

Let

$$\det \overline{G}(z_1,z_2) = \sum_{k=0}^{N_1}\sum_{l=0}^{N_1} a_{N_1-k,N_2-l} z_1^{-k} z_2^{-l} \tag{16}$$

It is assumed that i and j are bounded by some natural numbers L_1, L_2 that determine the degrees N_1, N_2.

From (14) and (11) it follows that

$$G^{-1}(z_1,z_2) = z_1^{-1} z_2^{-1} \overline{G}^{-1}(z_1,z_2) = z_1^{-1} z_2^{-1} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q} \tag{17}$$

and

$$\overline{G}^{-1}(z_1,z_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q} \tag{18}$$

where T_{pq} is defined by (6).

Theorem 2. Let (16) be the characteristic polynomial of the system (2). Then the matrices T_{kl} satisfy the equation

$$\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{kl} T_{kl} = 0 \tag{19}$$

Proof. From the definition of inverse matrix and (16), (18) we have

$$\mathrm{Adj}\overline{G}(z_1,z_2)=\left(\sum_{k=0}^{N_1}\sum_{l=0}^{N_2}a_{N_1-k,N_2-l}z_1^{-k}z_2^{-l}\right)\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}T_{pq}z_1^{-p}z_2^{-q}\right) \tag{20}$$

where $\mathrm{Adj}\overline{G}(z_1,z_2)$ is the adjoint matrix $\overline{G}(z_1,z_2)$.

Comparison of the coefficients at the same power $z_1^{-N_1}z_2^{-N_2}$ of the equality (20) yields (19) since the degrees of $\mathrm{Adj}\overline{G}(z_1,z_2)$ are less than N_1 and N_2. ■

Theorem 2 is an extension of the well-known classical Cayley-Hamilton theorem for the 2D fractional system (2).

3. Reachability and controllability to zero

Definition 2. The 2D fractional system (2) is called reachable at the point $(h,k)\in Z_+\times Z_+$ if and only if for zero boundary conditions (4)$(x=0,\ i\in Z_+,\ x_{0j}=0,\ j\in Z_+)$ and every vector $x_f\in\Re^n$ there exists a sequence of inputs $u_{ij}\in\Re^m$ for

$$(i,j)\in D_{hk}=\{(i,j)\in Z_+\times Z_+ : 0\le i\le h, 0\le j\le k, i+j\ne h+k\} \tag{21}$$

such that $x_{hk}=x_f$

Theorem 3. The 2D fractional system (2) is reachable at the point (h,k) if and only if the reachability matrix

$$R_{hk}=[M_0, M_1^1,\dots,M_{h2}^1, M_1^2,\dots,M_k^2, M_{11},\dots,M_{1k},M_{21},\dots,M_{hk}]$$

has full row rank, i.e.

$$\mathrm{rank}\ R_{hk}=n \tag{22}$$

where

$$\begin{aligned} &M_0=T_{h-1,k-1}B_0,\quad M_i^1=T_{h-i,k-1}B_1+T_{h-i-1,k-1}B_0,\ i=1,\dots,h\\ &M_j^2=T_{h-1,k-1}B_2+T_{h-1,k-j-1}B_0,\ j=1,\dots,k\\ &M_{ij}=T_{h-i-1,k-j-1}B_0+T_{h-i,k-j-1}B_1+T_{h-i-1,k-1}B_2,\ i=1,\dots,h,\ j=1,\dots,k \end{aligned} \tag{23}$$

Proof. Using the solution (5) for $i=h$, $j=k$ and zero boundary conditions we obtain

$$x_f=R_{hk}u(h,k) \tag{24}$$

where

$$u(h,k)=[u_{00}^T,u_{10}^T,\dots,u_{h0}^T,u_{01}^T,\dots,u_{0k}^T,u_{11}^T,\dots,u_{1k}^T,u_{21}^T,\dots,u_{h,k-1}^T]^T \tag{25}$$

and T denotes the transpose.

From (24) it follows that there exists a sequence $u_{ij}\in\Re^m$ for $(i,j)\in D_{hk}$ for every $x_f\in\Re^n$ if and only if the matrix R_{hk} contains n linearly independent columns equivalently the condition (22) is satisfied. ■

Example. Consider the 2D fractional system (2) with

$$A_0=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ A_1=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ A_2=\begin{bmatrix}1&0\\1&1\end{bmatrix}, \tag{26}$$

$$B_0=\begin{bmatrix}1\\0\end{bmatrix},\ B_1=\begin{bmatrix}0\\1\end{bmatrix},\ B_2=\begin{bmatrix}1\\1\end{bmatrix}$$

To check the reachability at the point $(h,k)=(1,1)$ of the system we use Theorem 3. From (23) and (22) we obtain

$$M_0=B_0=\begin{bmatrix}1\\0\end{bmatrix},\ M_1^1=B_1=\begin{bmatrix}0\\1\end{bmatrix},\ M_1^2=B_2=\begin{bmatrix}1\\1\end{bmatrix},\ M_{ij}=0$$

for $i\ge1, j\ge1$

$$R_{11}=[M_0, M_1^1, M_1^2]=\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix} \tag{27}$$

The matrix (27) has full row rank and by Theorem 3 the 2D fractional system (2) with (26) is reachable at the point $(1,1)$. The sequence of inputs steering the state of the system from zero boundary conditions to an arbitrary state $x_f\in\Re_+^2$ at the point $(1,1)$ is given by $\begin{bmatrix}u_{00}\\u_{10}\end{bmatrix}=x_f$ and $u_{01}=0$.

Note that the system is reachable at the point $(1,1)$ for any fractional order $\alpha, 0<\alpha<1$.

Definition 3. The 2D fractional system (2) is called controllable to zero at the point (h,k) if and only if for any nonzero boundary conditions (4) there exists a sequence of inputs $u_{ij}\in\Re^m$ for $(i,j)\in D_{hk}$ such that $x_{hk}=0$.

Theorem 4. The 2D fractional system (2) is controllable to zero at the point (h,k) if and only if the condition (22) is satisfied.

Proof. From (5) for $i=h, j=k$ and $x_{hk}=0$ we obtain

$$R_{hk}u(h,k)=-x_{bc}(h,k) \tag{28}$$

where

$$\begin{aligned} x_{lc}(i,j)=&\sum_{p=1}^{i}(T_{i-p,j-1}\overline{A}_1+T_{i-p-1,j-1}\overline{A}_0)x_{p0}+\\ &+\sum_{q=1}^{j}(T_{i-1,j-q}\overline{A}_2+T_{i-1,j-q-1}\overline{A}_0)x_{0q}+T_{i-1,j-1}\overline{A}_0x_{00}=0 \end{aligned}$$

The equation (28) has a solution $u(h,k)$ for arbitrary nonzero boundary conditions (4) if and only if the condition (22) is satisfied. ■

The considerations can be extended for $n-1<\alpha<n$, $n=1,2,\dots$

4. Concluding remarks

A new class of 2D fractional linear systems has been introduced. The notion of α order $(\alpha,\ 0<\alpha<1)$ 2D fractional difference has been proposed. The fractional 2D state equations of linear systems have been given and their solutions have been derived using the 2D Z transform. The classical Cayley-Hamilton theorem has been extended for the 2D fractional systems. Necessary and sufficient conditions have been established for the reachability and controllability to zero of the 2D fractional linear systems.

The considerations can be easily extended for fractional 2D linear systems with delays.

An extension of these considerations for fractional 2D continuous-time linear systems is an open problem.

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Appendix.

Justification of the definition 1.

It is well known that for a discrete function x_i the n-order difference is given by

$$\Delta_i^n x_i = \Delta^{n-1} x_i - \Delta^{n-1} x_{i-1} = \sum_{k=0}^{i} (-1)^k \binom{n}{k} x_{i-k} \quad \text{(A.1)}$$

$$n \in N = \{1,2,...\}, \quad i = Z_+\{0,1,...\}$$

where

$$\binom{n}{k} = \begin{cases} 1 & \text{for } k=0 \\ \dfrac{n!}{k!\,(n\text{-}k)!} \end{cases} \quad \text{(A.2)}$$

Using (A.1) for an 2D discrete function x_{ij} we obtain

$$\begin{aligned} \Delta_i^{n_1}\Delta_j^{n_2} x_{ij} &= \Delta_i^{n_2}\Delta_j^{n_1} x_{ij} = \sum_{k=0}^{i}(-1)^k \binom{n_1}{k} \Delta_i^{n_2} x_{i-k,j} = \\ &= \sum_{k=0}^{i}(-1)^k \binom{n_1}{k} \sum_{l=0}^{j}(-1)^l \binom{n_2}{l} x_{i-k,j} = \\ &= \sum_{l=0}^{j}(-1)^l \binom{n_2}{l} \sum_{k=0}^{i}(-1)^k \binom{n_1}{k} x_{i-k,j-l} = \\ &= \sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{k+l} \binom{n_1}{k}\binom{n_2}{l} x_{i-k,j-l} \end{aligned} \quad \text{(A.3)}$$

$$\text{for } n_1, n_2 \in N \text{ and } i,j \in Z_+$$

If $n_1 = n_2 = n$ then from (A.3) we obtain

$$\Delta^n x_{ij} = \Delta_i^n \Delta_j^n x_{ij} = \sum_{k=0}^{i}\sum_{j=0}^{j}(-1)^{k+l}\binom{n}{k}\binom{n}{l} x_{i-k,j-l} \quad \text{(A.4)}$$

Note that

$$\binom{n}{k}\binom{n}{l} = \begin{cases} 1 \text{ for } k=0 \text{ or/and } l=0 \\ \dfrac{n(n-1)...(n-k+1)n(n-1)...(n-l+1)}{k!\,l!} & \text{for } k+l>0 \end{cases} \quad \text{(A.5)}$$

is also well defined for $n=\alpha$ where α is any real number. Thus (A.4) can be used for defining the α order 2D fractional difference (1a).

Lemma. If $\alpha, 0<\alpha<1$ then

$$c_\alpha(k,l) > 0 \quad \text{for} \quad k+l>1 \ (k>0, l>0) \quad \text{(A.6)}$$

Proof. The proof will be accomplished by induction. The hypothesis is true for $k=2, l=1$ since

$$c_\alpha(2,1) = (-1)^3 \frac{\alpha^2(\alpha-1)}{2!\,1!} > 0 \qquad \text{for} \quad \alpha, 0<\alpha<1$$

The hypothesis is also true for $k=2, l=1$ since

$$c_\alpha(1,2) = (-1)^3 \frac{\alpha^2(\alpha-1)}{1!\,2!} > 0 \qquad \text{for} \quad \alpha, 0<\alpha<1$$

Assuming that the hypothesis is true for $k, l, k+l \geq 3$ we shall show that the hypothesis is also valid for the pair $(k+1, l)$ and $(k, l+1)$.

$$\begin{aligned} &c_\alpha(k+1,l) = \\ &= (-1)^{k+l+1}\frac{\alpha(\alpha-1)...(\alpha-k+2)\alpha(\alpha-1)...(\alpha-l+1)}{(k+1)!\,l!} = \\ &= -c_\alpha(k,l)\frac{\alpha-l+2}{l+1} > 0 \quad \text{for } 0<\alpha<1 \end{aligned}$$

Similarly,

$$\begin{aligned} &c_\alpha(k,l+1) = \\ &= (-1)^{k+l+1}\frac{\alpha(\alpha-1)...(\alpha-k+1)\alpha(\alpha-1)...(\alpha-l+2)}{k!\,(l+1)!} = \\ &= -c_\alpha(k,l)\frac{\alpha-l+2}{l+1} > 0 \quad \text{for } 0<\alpha<1 \end{aligned}$$

This completes the proof. ■

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