Fuzzy Relation-based Approximation Techniques in Supporting Medical Diagnosis

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Abstract:
In this paper we present an application of fuzzy approximation operators in supporting medical diagnosis. These operators are compositions of fuzzy modal operators. The underlying idea is based on the observation that approximations of fuzzy sets may be viewed as intuitionistic fuzzy sets. Reasoning scheme is determined by distances between intuitionistic fuzzy sets proposed by Szmidt and Kacprzyk.

Keywords: *Fuzzy modal operators, Fuzzy set approximations, Intuitionistic fuzzy sets, Medical diagnosis*

1. Introduction

In real-life problems we deal with information which is usually incomplete. The reasons are generally twofold. First, it follows from the fact that only partial data about the problem under consideration can be obtained. Second, the available data are often given in an imprecise form, for example when expressed using linguistic terms like "*quite good*" or "*rather cold*". Therefore, new information derived from incomplete data is in general uncertain.

In many applications the available information has a form of a set of objects and a set of their properties. Formal methods of analysis of such information were extensively developed within rough set theory (see, for example, Demri and Orłowska [5] and Orłowska [8]). While descriptions of objects are explicit information, relationships between objects/properties are new data that can be derived and constitute implicit information about domains in discourse. Such relationships are represented by *information relations*. A typical example of an information relation is an indiscernibility relation: two objects are indiscernible whenever they have the same selected properties. Approximation techniques, usually based on modal (or modal-like) operators, are applied in reasoning schemes.

Fuzzy set theory, originally introduced by Zadeh [33], offers a variety of methods for representing and processing imprecise (or vague, fuzzy) information. Therefore, when information of such a kind is admitted, fuzzy generalizations of traditional techniques are to be applied. Fuzzy information relations were widely investigated by Radzikowska and Kerre [19, 25, 27]. Logical systems capable to reason about these relations were considered by Radzikowska [14]. Comprehensive expositions of logical and algebraic aspects of information relations and knowledge and approximation operators were presented by Orłowska, Radzikowska, and Rewitzky [9].

Fuzzy sets allow for representation of graded information in the sense that degrees of memberships are given, yet one is unaware to what extend non-membership refers. For instance, if we know that a patient p suffers from pneumonia up to the degree 0.7, we can only say that this disease is excluded for p at most to the degree 0.3. Atanassov [1, 2] generalized fuzzy sets by providing two parameters for each element of the universe in discourse: the degrees of membership and the degree of non-membership. This allows us for stating that, e.g., p suffers from pneumonia up to the degree 0.7 and pneumonia is excluded for p up to the degree 0.1, thus 0.2 is the hesitation degree i.e., our lack of knowledge. In consequence, we obtain a more flexible tool for representing vague information.

In this paper we present an application of relation-based approximation techniques to medical diagnosis problem. Assume that we are given a set P of patients, a set D of some diseases, and a set S of symptoms of diseases from D. Each patient $p \in P$ is characterized by symptoms $s \in S$, and each disease $d \in D$ is described in terms of its symptoms $s \in S$. Our aim is to derive the proper medical diagnosis for each patient. For this purpose it is necessary to use information obtained from medical tests made for patients as well as medical knowledge about diseases. Medical knowledge actually occurs in two forms: as an *explicit* information given in the form of descriptions of diseases, and *implicit* knowledge that can be derived from these descriptions. Our methodology involves approximation methods that allow us to determine to what extend particular patient (resp. disease) *at least* and *at most* shows (resp. is characterized by) particular symptoms. Having applied these techniques we determine two intuitionistic fuzzy relations representing descriptions of patients and diseases, respectively, in terms of their symptoms. Following the idea proposed by Szmidt and Kacprzyk [29, 32], medical diagnosis is determined by distances between intuitionistic fuzzy sets.

The paper is organized as follows. In Section 2 we recall basic notions of fuzzy sets, fuzzy relations, fuzzy logical connectives and intuitionistic fuzzy sets. Next, in Section 3, we present fuzzy approximation operators based on fuzzy relations. An application of these operations for supporting medical diagnosis is discussed in Section 4. Concluding remarks complete the paper.

2. Preliminaries

In this section we recall basic notions of fuzzy set theory which are used in our presentation.

2.1. Fuzzy Sets

Let X be a non-empty domain. A *fuzzy set in* X is any mapping $F : X \to [0,1]$. For every $x \in X$, $F(x)$ is the degree to which x belongs to F. Given two fuzzy set $A, B \in \mathcal{F}(X)$,

- A is (totally) included in B, written $A \subseteq B$, if $A(x) \leqslant B(x)$ for every $x \in X$;
- A is (totally) equal to B, written $A = B$, if $A(x) = B(x)$ for every $x \in X$.

The family of all fuzzy sets in X will be denoted by $\mathcal{F}(X)$.

A *fuzzy relation in* X *and* Y is a fuzzy set in $X \times Y$. For $x \in X$ and for $y \in Y$, $R(x, y)$ is the degree to which x is R-related with y. A fuzzy relation R in X and Y is called *crisp* if $R(x, y) \in \{0, 1\}$ for all $x \in X$ and for all $y \in Y$. The family of all fuzzy relations in X and Y will be written $\mathcal{R}(X, Y)$. For $R \in \mathcal{R}(X, Y)$, the converse relation $R^{-1} \in \mathcal{R}(Y, X)$ is defined as $R^{-1}(y, x) = R(x, y)$. For every $R \in \mathcal{R}(X, Y)$ and for every $x \in X$, we write xR to denote the fuzzy set in Y defined as $(xR)(y) = R(x, y)$. Analogously, for any $y \in Y$, $Ry \in \mathcal{F}(X)$ is defined as $(Ry)(x) = R(x, y)$. A fuzzy relation $R \in \mathcal{R}(X, X)$ is a fuzzy relation on X.

2.2. Fuzzy Logical Connectives

Fuzzy logical connectives are generalizations of logical connectives of classical logic. *Triangular norms* generalize classical conjunction. Specifically, a triangular norm (t-norm, for short) is a mapping $\otimes : [0,1]^2 \to [0,1]$, commutative ($x \otimes y = y \otimes x$, $x, y \in [0,1]$), associative ($x \otimes (y \otimes z) = (x \otimes y) \otimes z$, $x, y, z \in [0,1]$), increasing in both arguments ($x \leqslant z$ implies $x \otimes y \leqslant z \otimes y$ and $y \otimes x \leqslant y \otimes z$, $x, y, z \in [0,1]$), and satisfying the boundary condition $x \otimes 1 = x$ for every $x \in [0,1]$. The most popular t-norms are:

- the standard t-norm (the largest t-norm)

$$x \otimes_Z y = \min(x, y)$$

- the product operation

$$x \otimes_P y = xy$$

- the Łukasiewicz t-norm

$$x \otimes_L y = \max(0, x + y - 1).$$

A t-norm $\otimes$ is left-continuous whenever it is left-continuous on both arguments. For the extended studies on t-norms we refer a reader to Klement, Mesiar and Pap [7].

A *fuzzy implication* is a $[0,1]^2 - [0,1]$ map $\to$ with decreasing 1^{st} and increasing 2^{nd} partial mappings ($x \leqslant z$ implies $z \to y \leqslant x \to y$ and $y \to x \leqslant y \to z$ for all $x, y, z \in [0,1]$) and satisfying $1 \to 1 = 0 \to 0 = 0 \to 1 = 1$ and $1 \to 0 = 0$. The most popular fuzzy implications are

- the Kleene-Dienes implication

$$x \to_{KD} y = \max(1 - x, y)$$

- the Łukasiewicz implication

$$x \to_L y = \min(1, 1 - x + y)$$

- the Gödel implication

$$x \to_G y = \begin{cases} 1 & \text{for } x \leqslant y \\ y & \text{elsewhere.} \end{cases}$$

A special class of fuzzy implications are *residual implications*: given a left-continuous t-norm $\otimes$ its residual implication (also called the residuum of $\otimes$) is defined for all $x, y \in [0,1]$,

$$x \to y = \sup\{z \in [0,1] : x \otimes z \leqslant y\}.$$

The Łukasiewicz and the Gödel implications are examples of residual implications based on $\otimes_L$ and $\otimes_M$, respectively, while Kleene-Dienes implication is not a residual one. Fuzzy implications were extensively investigated by Baczyński and Jayaram [3].

A *fuzzy negation* is a mapping $\neg : [0,1] \to [0,1]$, decreasing and satisfying $\neg 0 = 1$ and $\neg 1 = 0$. The standard fuzzy negation is $\neg_S x = 1 - x$ for every $x \in [0,1]$. Residual implications lead to fuzzy negations: $\neg x = x \to 0$. The Łukasiewicz implication induces the standard fuzzy negation, that is $\neg_L = \neg_S$, and the Gödel implication induces the fuzzy negation $\neg_G x = 0$ for $x \neq 1$ and $\neg_G 1 = 0$.

Given a fuzzy set $A \in \mathcal{F}(X)$ and a fuzzy negation $\neg$, we write $\neg A$ to denote the $\neg$-complementation of A, that is the fuzzy set in X defined for every $x \in X$, $(\neg A)(x) = \neg A(x)$.

2.3. Intuitionistic Fuzzy Sets

Now, let us recall basic notions of intuitionistic fuzzy set theory (see Atanassov [1]). Let a non-empty domain X be given. An *intuitionistic fuzzy set in* X is given by $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$, where $\mu_A \in \mathcal{F}(X)$ and $\nu_A \in \mathcal{F}(X)$ are called a membership and a non-membership function, respectively, and satisfy $\mu_A(x) + \nu_A(x) \leqslant 1$ for every $x \in X$. The value $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, $x \in X$, is called a *hesitation margin* which reflects the lack of knowledge of membership or non-membership of x to A. The family of all intuitionistic fuzzy sets in X will be denoted by $\mathcal{IF}(X)$.

Clearly, any fuzzy sets A in X is a specific intuitionistic fuzzy set $A = \{(x, \mu_A(x), 1 - \mu_A(x)) : x \in X\}$. An intuitionistic fuzzy relation in X and Y is an intuitionistic fuzzy set in $X \times Y$, i.e., it is given by $R = \{((x, y), \mu_R(x, y), \nu_R(x, y)) : x \in X \text{ and } y \in Y\}$ with $\mu_R, \nu_R \in \mathcal{R}(X, Y)$ satisfying $\mu_R(x, y) + \nu_R(x, y) \leq 1$ for all $x \in X$ and $y \in Y$. Accordingly, $\mu_R(x, y)$ is the degree to which $x \in X$ is R–related with $y \in Y$, $\nu_R(x, y)$ is the degree to which x and y are not R-related, and $\pi_R(x, y) = 1 - \mu_R(x, y) - \nu_R(x, y)$ is a hesitation margin. For the extensive studies of the theory of intuitionistic fuzzy sets we refer to [2].

Traditionally, a distance between two intuitionistic fuzzy sets in X is defined with respect to two parameters, that is the degrees of membership and the degrees of non-membership. The drawback of this approach was pointed out by Szmidt and Kacprzyk in [29] and a novel definition was proposed where all three parameters, that is including hesitation regions, are taken into account. More specifically, let $X = \{x_1, \dots, x_n\}$ and let $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ and $B = \{(x, \mu_B(x), \nu_B(x)) : x \in X\}$ be two intuitionistic fuzzy sets in X. Then

- the normalized Hamming distance between A and B:

$$\delta_H(A, B) = \frac{1}{2n} \sum_{i=1}^{n} d_i \tag{1}$$

where

$d_i = |\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|$.

- the normalized Euclidean distance between A and B:

$$\delta_E(A, B) = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} e_i} \tag{2}$$

where

$e_i = (\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2$.

3. Relation-based Fuzzy Set Approximations

Let X and Y be two non-empty universes and let $R \subseteq X \times Y$ be a relation on X and Y. Intuitively, X may be viewed as a set of objects, Y is treated as a set of their properties, and for every $x \in X$ and for every $y \in Y$, xRy states that an object x has the property y. Note that any set $A \subseteq X$ may be viewed as a representation of an expert decision concerning objects from X, or as a representation of some feature (not necessarily from Y) characterizing particular objects from the set X. Analogously, any set $B \subseteq Y$ may represent characterization of some object (not necessarily from X) in terms of properties from the set Y.

Any relation $R \subseteq X \times Y$ allows us to derive some implicit information about objects from X, and properties from Y. Specifically, we can infer about links between objects (resp. properties) basing on their properties (resp. objects having these properties). In the terminology well-known in rough set theory (see, e.g., Demri and Orłowska [5]) these links are formalized by *information relations*. Here let us recall two of such relations:

- *compatibility*:
  - for objects: objects x_1 and x_2 are compatible if they share some common property $y \in Y$;
  - for properties: properties y_1 and y_2 are compatible if some object $x \in X$ has both properties.
- *relevance* (also called inclusion, or forward inclusion)
  - for objects: an object x_1 is relevant to an object x_2 if all properties of x_1 are also properties of x_2;
  - for properties: a property y_1 is relevant to a property y_2 if all objects having the property y_1 have also the property y_2.

In the following we will not indicate directly whether compatibility (resp. relevance) refers to objects or properties since it will clearly follow from the context they are used in.

Example 3.1 Let us consider a set P of four people: *Al*, *Bob*, *Joe*, and *Ted* and a set S of five symptoms of diseases these patients suffer from: *Temperature*, *Headache*, *Stomach pain*, *Cough*, and *Chest pain*. Tab. 1 represents characterization of the patients in terms of their symptoms given by a binary (crisp) relation.

Note that *Al* and *Joe* are compatible since they both have temperature, while *Temperature* and *Headache* are compatible since *Al* shows both symptoms. Moreover, *Joe* is relevant to *Al* and *Cough* is relevant to *Temperature*. □

Information, as given in Example 3.1, although sometimes useful, in many real-life problems is practically meaningless. In particular, it is unknown how strong Al's headache is, whether indeed nobody shows chest pain, or may be some patients suffer from a very slight one, etc. If medical diagnosis is to be determined, we essentially need to know to what extend patients show particular symptoms. These leads us to fuzzy structures which are commonly used for representation of graded information. In Tab. 2 a fuzzy relation $R \in \mathcal{R}(P, S)$ shows to what degree particular patients show specific symptoms.

When imprecise data are involved, we actually have fuzzy information relations, in particular a fuzzy compatibility and a fuzzy relevance. Fuzzy information relations were extensively investigated by Radzikowska and Kerre [16, 17, 19, 20, 22, 25].

In order to infer about relationships between object/properties in the environment of fuzzy information, fuzzy modal operators are useful. These operators were investigated and widely discussed by Radzikowska and Kerre [10–12, 24, 26, 27]. Let us recall some basic facts. Given a t-norm $\otimes$ and its residual implication $\rightarrow$, the following two $\mathcal{F}(Y) - \mathcal{F}(X)$ operators are defined for every fuzzy relation $R \in \mathcal{R}(X, Y)$, for any fuzzy set $A \in \mathcal{F}(Y)$, and for every $x \in X$,

$$([R]_\otimes A)(x) = \inf_{y \in Y} (R(x, y) \rightarrow A(y)) \tag{3}$$

$$(\langle R \rangle_\otimes A)(x) = \sup_{y \in Y} (R(x, y) \otimes A(y)) \tag{4}$$

The operators (3) and (4) are called *fuzzy necessity* and *fuzzy possibility*, respectively. Assume that R is a fuzzy relation on X, for instance, a fuzzy similarity relation which reflects similarities of object determined

Tab. 1. Patients and their symptoms

	Temperature	*Headache*	*Stomach pain*	*Cough*	*Chest pain*
Al	1	1	0	1	0
Bob	0	0	1	0	0
Joe	1	1	0	0	0
Ted	1	0	0	1	0

Tab. 2. Symptoms characteristic for the patients considered

R	*Temperature*	*Headache*	*Stomach pain*	*Cough*	*Chest pain*
Al	0.8	0.6	0.2	0.6	0.1
Bob	0.0	0.4	0.6	0.1	0.1
Joe	0.8	0.8	0.0	0.2	0.0
Ted	0.6	0.5	0.3	0.7	0.3

by their properties. Then (3) and (4) are *fuzzy lower* and *fuzzy upper rough approximation operators* extensively studied by Radzikowska and Kerre [13, 18, 21, 23].

The intuitive meaning of (3) and (4) is the following: given $A \in \mathcal{F}(Y)$ representing characterization of some object, and $x \in X$,

- $([R]_\otimes A)(x)$ is the degree to which the object x is relevant to A;
- $(\langle R\rangle_\otimes A)(x)$ is the degree to which the object x is compatible with A.

Analogously, taking $B \in \mathcal{F}(X)$ representing an expert decision, for any $y \in Y$, $([R^{-1}]_\otimes B)(y)$ is the degree to which the property y is relevant to the decision B and $(\langle R^{-1}\rangle_\otimes B)(y)$ is the degree to which y is compatible with B. Now, basing on the above operators let us define two mappings $\blacktriangle^R_\otimes, \blacktriangledown^R_\otimes : \mathcal{F}(Y) \to \mathcal{F}(Y)$ for every $A \in \mathcal{F}(Y)$,

$$\blacktriangle^R_\otimes A = \langle R^{-1}\rangle_\otimes [R]_\otimes A \tag{5}$$

$$\blacktriangledown^R_\otimes A = [R^{-1}]_\otimes \langle R\rangle_\otimes A. \tag{6}$$

These operators are fuzzy generalizations of the respective operators investigated by Düntsch and Gediga in [6]. Intuitively, for any $A \in \mathcal{F}(Y)$ and for any $y \in Y$,

- $(\blacktriangle^R_\otimes A)(y)$ is the degree to which some object characterized by the property y is relevant to the object A;
- $(\blacktriangledown^R_\otimes A)(x)$ is the degree to which all objects characterized by the property y are compatible with the object A.

Radzikowska [12] showed that for every $R \in \mathcal{R}(X,Y)$ and for all $A, B \in \mathcal{F}(Y)$,

(P1) $A \subseteq B$ implies $\blacktriangle^R_\otimes A \subseteq \blacktriangle_\otimes B$ and $\blacktriangledown^R_\otimes A \subseteq \blacktriangledown^R_\otimes B$,

(P2) $\blacktriangle^R_\otimes A \subseteq A \subseteq \blacktriangledown^R_\otimes A$,

(P1) states that both operators are monotone and, due to (P2), they work as approximation operators: $\blacktriangle^R_\otimes A$ is a *lower bound* of A, whereas $\blacktriangledown^R_\otimes A$ is an *upper bound* of A. Therefore, for every $y \in Y$, $(\blacktriangle^R_\otimes A)(y)$ can be viewed as the degree to which y *at least* (certainly) belongs to A, and $(\blacktriangledown^R_\otimes A)(y)$ can be interpreted as the degree to which y *at most* (possibly) belongs to A. Hence, the value $\neg_\otimes \blacktriangledown^R_\otimes A(y)$, where $\neg_\otimes$ is the fuzzy negation induced by $\otimes$, is the degree to which y certainly does not belong to A. For any $A \in \mathcal{F}(Y)$, the pair $(\blacktriangle^R_\otimes A), \blacktriangledown^R_\otimes A)$ will be referred to $(\blacktriangle^R_\otimes, \blacktriangledown^R_\otimes)$-*approximation of A with respect to R and $\otimes$.*

Remark 3.1

1) *Note that for any $y \in Y$, we have $\blacktriangle^R_\otimes(xR) = xR$. In general, however, $\blacktriangledown^R_\otimes(xR) \neq xR$.*
2) *Let $R \in \mathcal{R}(X,Y)$ and let $A \in \mathcal{F}(Y)$ be given. For an arbitrary left-continuous t-norm $\otimes$,*

$$\blacktriangle^R_\otimes A(y) + \neg_\otimes \blacktriangledown^R_\otimes A(y) \nleqslant 1.$$

Let $A \in \mathcal{F}(Y)$ and let $\blacktriangle^R_L A$ and $\blacktriangledown^R_L A$ be its lower and upper bounds, respectively, determined by the Łukasiewicz t-norm $\otimes_L$. Since the negation $\neg_L$ induced by the Łukasiewicz t-norm is the standard fuzzy negation $\neg_s$, the following condition holds for every $y \in Y$:

$$(\blacktriangle^R_L A)(y) + (\neg_L \blacktriangledown^R_L A)(y) \leqslant 1.$$

Consequently, the $(\blacktriangle^R_L, \blacktriangledown^R_L)$-approximation of A uniquely determines an intuitionistic fuzzy set in Y as the following observation states.

Observation 3.1 *For every fuzzy relation $R \in \mathcal{R}(X,Y)$, the $(\blacktriangle^R_L, \blacktriangledown^R_L)$-approximation of any fuzzy set $A \in \mathcal{F}(Y)$ determines an intuitionistic fuzzy set $A' = \{(y, \mu_{A'}(y), \nu_{A'}(y)) : y \in Y\}$ in Y given by:*

$$\mu_{A'}(y) = \blacktriangle^R_L A(y)$$

$$\nu_{A'}(y) = 1 - \blacktriangledown^R_L A(y).$$

This idea is the basis for the medical diagnosis problem presented in the next section.

Finally, it is worth noting that Łukasiewicz logical connectives are very useful in fuzzy generalizations of many structures. Radzikowska and Kerre [18] showed that these fuzzy connectives are the best ones for fuzzy generalization of traditional (crisp) rough sets.

4. Medical Diagnosis Using Fuzzy Relation-based Approximations

Let us consider a set P of patients, a set S of symptoms of some diseases, and a set D of medical diagnosis. On the basis of medical knowledge each diagnosis is characterized by particular symptoms. Also, having made some medical tests each patient is described by symptoms he shows. Our aim is to determine a proper diagnosis for each patient.

As noted before, a fuzzy approach is highly justified for this problem. Szmidt and Kacprzyk [30, 32] assumed that a given information is represented by intuitionistic fuzzy relations. Similar representation was earlier given by De, Biswas, and Roy [4]. In these approaches it is required to know to what extend particular symptoms characterize given diagnosis (resp. patients) as well as to what extent symptoms they do not characterize given diagnosis (resp. patients). Here we assume that the available information is more restricted: all we know about symptoms is to what extent they characterize diagnosis (resp. patients). Then we have two fuzzy relations: a relation $R \in \mathcal{R}(P, S)$ which provides descriptions of particular patients in terms of symptoms they show and a relation $Q \in \mathcal{R}(D, S)$ which describes particular diagnosis in terms of their characteristic symptoms. For a set P of patients and a set S of symptoms given in Section 3, and a set $D = \{$*Viral fever*, *Malaria*, *Typhoid*, *Stomach problem*, *Chest problem*$\}$ of diagnosis, examples of fuzzy relation R and Q are presented in Tab. 2 and Tab. 3, respectively. Similar data were presented by Szmidt and Kacprzyk [30] and De, Biswas, and Roy [4].

Clearly, in order to make a proper diagnosis one needs the possibly broadest medical knowledge. Note that a relation Q represents only partial medical knowledge which may be referred to as explicit knowledge. New information that are derived from Q constitutes an implicit knowledge which should be taken into account in the process of medical diagnosing.

The simplest solution of our problem is based on distances of fuzzy sets. For each patient $p \in P$ we consider a distance between his/her description pR and characterization of particular diagnosis dQ. The proper diagnosis is pointed out by the shortest distance. However, this method does not take into account implicit medical knowledge that can be derived from a relation Q which, in turn, may lead to highly doubtful results. Another solution, originally proposed by Sanchez ([28] and later on developed by De, Biswas, and Roy [4]), is based on a composition of fuzzy relations R and Q. Namely, we determine the fuzzy relation $T = R \circ Q^{-1} \in \mathcal{R}(P, D)$ defined for every $p \in P$ and for every $d \in D$,

$$T(p, d) = \sup_{s \in S} \min(R(p, s), Q^{-1}(d, s)).$$

In the context of intuitionistic fuzzy sets De, Biswas, and Roy [4], as well as Szmidt and Kacprzyk [32], pointed out that this method has an essential drawback since it prefers dominating symptoms which could make the diagnosis incorrect.

Here we present another approach using approximation operators (5) and (6) with the underlying relation Q. This way the approximations are determined by medical knowledge, both explicit and implicit. Firstly, for each patient p we approximate his/her characterization in terms of symptoms, i.e., the fuzzy set pR. Next, in analogous way each diagnosis description Qd is approximated. Using Observation 3.1, two IF-relations are then obtained: patient-symptoms relation and symptom-diagnosis relation. Finally, following the idea proposed by Szmidt and Kacprzyk [30–32], we calculate distances between the IF-set style description of patients and the IF-set style description of diagnosis. For each patient the shortest distance points out his proper medical diagnosis.

Concretely, as in Radzikowska [15], the following procedure is proceeded.

Step 1: For each patient $p \in P$, approximate his symptoms with respect to the diagnosis-symptom relation Q: determine the $(\blacktriangle^Q_L, \blacktriangledown^Q_L)$-approximation of pR.

Step 2: For each diagnose $d \in D$, approximate its symptoms with respect to the diagnosis-symptom characterization Q: calculate $(\blacktriangle^Q_L, \blacktriangledown^Q_L)$-approximation of dQ.

Step 3: Determine IF-relations R' (patient–symptom) and Q' (diagnosis-symptom) on the basis of approximations obtained in Step 1 and 2: calculate

$$\begin{aligned}
\mu_{R'}(p, s) &= \blacktriangle^Q_L(pR) \\
\nu_{R'}(p, s) &= 1 - \blacktriangledown^Q_L(pR) \\
\pi_{R'}(p, s) &= \blacktriangledown^Q_L(pR) - \blacktriangle^Q_L(pR)
\end{aligned}$$

and

$$\begin{aligned}
\mu_{Q'}(d, s) &= \blacktriangle^Q_L(sQ) \\
\nu_{Q'}(d, s) &= 1 - \blacktriangledown^Q_L(sQ) \\
\pi_{Q'}(d, s) &= \blacktriangledown^Q_L(sQ) - \blacktriangle^Q_L(sQ),
\end{aligned}$$

respectively.

Step 4: For each $p \in P$ and for each diagnosis $d \in D$, calculate distance between pR' and dQ' – the lowest distance points out the proper diagnosis.

Tab. 4 presents lower and upper bounds of pR for every patient $p \in P$. Note that due to medical tests *Al* suffers from cough up to the degree 0.6 (see Tab. 2). Having taken into account his other symptoms and medical knowledge about specificity of cough for particular diseases (Tab. 3), it was estimated that *Al* has cough *at least* to the degree 0.6 and *at most* to the degree 0.8. Using linguistic terms one can say that his cough is estimated between *rather strong* and *strong*. Also, it turns out that his headache and stomach pain totally coincide with what was established by medical tests (Tab. 4), that is, he shows both symptoms at least and at most up to the same degree.

Tab. 3. Symptoms characteristic for the diagnosis considered

Q	*Temperature*	*Headache*	*stomach pain*	*Cough*	*Chest pain*
Viral fever	0.4	0.3	0.1	0.4	0.1
Malaria	0.7	0.1	0.0	0.7	0.1
Typhoid	0.3	0.6	0.2	0.2	0.1
Stomach problem	0.1	0.2	0.8	0.2	0.2
Chest problem	0.1	0.0	0.2	0.2	0.8

Tab. 4. Approximated symptoms characteristic for the patients

	Temperature	*Headache*	*Stomach pain*	*Cough*	*Chest pain*
Al	(0.6,0.8)	(0.6,0.6)	(0.2,0.2)	(0.6,0.8)	(0.1,0.2)
Bob	(0.0,0.3)	(0.3,0.4)	(0.6,0.8)	(0.0,0.3)	(0.1,0.2)
Joe	(0.2,0.8)	(0.4,0.8)	(0.0,0.2)	(0.2,0.8)	(0.0,0.2)
Ted	(0.6,0.7)	(0.5,0.5)	(0.3,0.3)	(0.6,0.7)	(0.3,0.3)

Tab. 5. Approximated symptoms characteristic for the diagnosis

	Temperature	*Headache*	*Stomach pain*	*Cough*	*Chest pain*
Viral fever	(0.4,0.4)	(0.3,0.4)	(0.1,0.2)	(0.4,0.4)	(0.1,0.2)
Malaria	(0.7,0.7)	(0.2,0.4)	(0.6,0.7)	(0.0,0.2)	(0.7,0.7)
Typhoid	(0.3,0.3)	(0.6,0.7)	(0.2,0.2)	(0.2,0.3)	(0.1,0.2)
Stomach problem	(0.1,0.3)	(0.2,0.4)	(0.8,0.8)	(0.2,0.3)	(0.2,0.2)
Chest problem	(0.1,0.3)	(0.0,0.4)	(0.2,0.2)	(0.2,0.3)	(0.8,0.8)

Tab. 6. Patient-symptom intuitionistic fuzzy relation

R'	*Temperature*	*Headache*	*Stomach pain*	*Cough*	*Chest pain*
Al	(0.6,0.2,0.2)	(0.6,0.4,0,0)	(0.2,0.8,0.0)	(0.6,0.2,0.2)	(0.1,0.8,0,1)
Bob	(0.0,0.7,0.3)	(0.3,0.6,0.1)	(0.6,0.2,0.2)	(0.0,0.7,0.3)	(0.1,0.8,0.1)
Joe	(0.2,0.2,0.6)	(0.4,0.2,0.4)	(0.0,0.8,0.2)	(0.2,0.2,0.6)	(0.0,0.8,0.2)
Ted	(0.6,0.3,0.1)	(0.5,0.5,0.0)	(0.3,0.7.0.0)	(0.6,0.3,0.1)	(0.3,0.7,0.0)

Tab. 7. Diagnosis-symptom intuitionistic fuzzy relation

Q'	*Temperature*	*Headache*	*Stomach pain*	*Cough*	*Chest pain*
Viral fever	(0.4,0.6,0)	(0.3,0.6,0.1)	(0.1,0.8,0.1)	(0.4,0.6,0.0)	(0.1,0.8,0.1)
Malaria	(0.7,0.3,0.0)	(0.2,0.6,0.2)	(0.0,0.8,0.2)	(0.7,0.3,0.0)	(0.1,0.8,0.1)
Typhoid	(0.3,0.7,0.0)	(0.6,0.3,0.1)	(0.2,0.8,0.0)	(0.2,0.7,0.1)	(0.1,0.8,0.1)
Stomach problem	(0.1,0.7,0.2)	(0.2,0.6,0.2)	(0.8,0.2,0.0)	(0.2,0.7,0.1)	(0.2,0.8,0.0)
Chest problem	(0.1,0.7,0.2)	(0.0,0.6,0.4)	(0.2,0.8,0.0)	(0.2,0.7,0.1)	(0.8,0.2,0.0)

Tab. 8. The normalized Hamming distances for patients from the possible diagnosis

	Viral fever	*Malaria*	*Typhoid*	*Stomach problem*	*Chest problem*
Al	0.24	**0.20**	0.22	0.42	0.46
Bob	0.28	0.42	0.28	**0.14**	0.38
Joe	0.36	0.34	**0.32**	0.48	0.48
Ted	0.24	**0.20**	0.26	0.36	0.40

Next, characteristics of particular diagnosis given in Tab. 3 are approximated using medical knowledge represented in the fuzzy relation Q. The results are presented in Tab. 5. In particular, for *Temperature*, its lower and upper bounds coincide for *Viral fever*, *Malaria*, and *Typhoid*, so the relation Q itself precisely characterizes this symptom for these diseases. For the two remaining diseases, however, *Temperature* can be stated only approximately in view of the derived information.

On the basis of approximations given in Tab. 4 and Tab. 5, two intuitionistic fuzzy relations are calculated and the results are given in Tab. 6 and Tab. 7, respectively. For example, *Joe* has a headache up to the degree

Tab. 9. The normalized Euclidean distances for patients from the possible diagnosis

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	0.064	**0.044**	0.082	0.190	0.234
Bob	0.114	0.222	0.100	**0.022**	0.170
Joe	0.142	0.150	**0.136**	0.228	0.236
Ted	0.046	**0.038**	0.072	0.140	0.168

0.4 and at the same time this symptom is slightly excluded (to the degree 0.2), so it is unknown whether he actually suffers from this pain up to the degree 0.4. Similarly, *Headache* is not a characteristic symptom for *Chest problem*, but it is excluded for this diagnosis only up to the degree 0.6 – it is then undetermined whether this symptom is specific for this diagnosis up to the degree 0.4.

Now, taking into account data from Tab. 6 and Tab. 7 we calculate distances between intuitionistic fuzzy set pR' and dQ' for every $p \in P$ and for every $d \in D$. For the normalized Hamming distance the results are shown in Tab. 8. The shortest distance points out the proper diagnosis. Namely, *Al* and *Ted* suffer from malaria, *Bob* from stomach problem, and *Joe* has typhoid. For the normalized Euclidean distance the results are similar as shown in Tab. 9.

5. Concluding Remarks

In this paper we have shown an application of fuzzy approximation operators in supporting medical diagnosis. These operators are compositions of fuzzy necessity and fuzzy possibility modal operators well-known in fuzzy modal logics. Our approach is based on the observation that approximations of fuzzy sets lead to intuitionistic fuzzy sets. Then, given two fuzzy relations representing characterizations of patients and diseases, respectively, in terms of their symptoms, we have obtained two respective intuitionistic fuzzy relations. Following the idea proposed by Szmidt and Kacprzyk [30], proper medical diagnosis are determined basing on distances between intuitionistic fuzzy sets.

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