

# COMPARISON OF ALGORITHMS FOR DECISION MAKING PROBLEMS AND PRESERVATION OF $\alpha$ -PROPERTIES OF FUZZY RELATIONS IN AGGREGATION PROCESS

Submitted: 25<sup>th</sup> January 2016; accepted: 16<sup>th</sup> May 2016

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DOI: 10.14313/JAMRIS\_2-2016/13

## Abstract:

In the paper the problem of preservation of properties of fuzzy relations during aggregation process is considered. It means that properties of fuzzy relations  $R_1, \dots, R_n$  on a set  $X$  are compared with properties of the aggregated fuzzy relation  $R_F = F(R_1, \dots, R_n)$ , where  $F$  is a function of the type  $F : [0, 1]^n \rightarrow [0, 1]$ . There are discussed  $\alpha$ -properties (which may be called graded properties - to some grade  $\alpha$ ) as reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness and transitivity, where  $\alpha \in [0, 1]$ . Fuzzy relations with a given graded property are analyzed (there may be diverse grades of the same property) and the obtained grade of the aggregated fuzzy relation is provided. There is also discussed the „converse” problem. Namely, relation  $R_F = F(R_1, \dots, R_n)$  is assumed to have a graded property and the properties of relations  $R_1, \dots, R_n$  are examined (possibly with some assumptions on  $F$ ). Presented here considerations have possible applications in decision making algorithms. This is why interpretation of the considered graded properties and possible potential in decision making is presented.

**Keywords:** decision making algorithms, fuzzy relations, properties of fuzzy relations, aggregation functions

## 1. Introduction

Since Zadeh has introduced definition of fuzzy relations [38], [39], the theory of them was developed by several authors. Thanks to the „fuzzy environment” we may discuss diverse types of fuzzy relation properties. For example, graded properties of fuzzy relations were observed in [23] and  $\alpha$ -properties were introduced in [10]. These properties may be understood as properties to some grade  $\alpha$ , where  $\alpha \in [0, 1]$ .

Aggregation functions, including means [24], are now widely investigated and there are a few monographs devoted to this topic, e.g. [2], [7], [22]. Aggregation is a fundamental process in multicriteria decision making and in other scientific disciplines where the fusion of different pieces of information for obtaining the final result is important. For example, in the multicriteria decision making a finite set of alternatives  $X = \{x_1, \dots, x_m\}$  and a finite set of criteria on the base of which the alternatives are evaluated  $K = \{k_1, \dots, k_n\}$  may be considered. Fuzzy relations  $R_1, \dots, R_n$  on a set  $X$  corresponding to each criterion are provided. With the use of a function  $F$  the aggregated fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is obtained and it is supposed to help decision makers to make

up their mind. It is useful to know which properties of fuzzy relations  $R_1, \dots, R_n$  are transposed to the relation  $R$ . There are several works contributed to the problem of preservation of properties of fuzzy relations during aggregation process, e.g. [21], [31], [32], [34].

In this paper the problem of preservation of graded properties of fuzzy relations (cf. [14], [16], [18]) is examined. A finite number of fuzzy relations having a given graded property is considered (there can be diverse grades of the same property) and the obtained grade of the aggregated fuzzy relation is provided. There are discussed several graded properties: reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness and transitivity. There is also considered another problem. Namely, relation  $R_F = F(R_1, \dots, R_n)$  is assumed to have a graded property and relations  $R_1, \dots, R_n$  are examined whether they have the same property. Appropriate assumptions on  $F$  to fulfill the required property are proposed. Presented in this paper results may have applications in decision making problems what is more widely described in Section 3. Moreover, the interpretation of the graded properties in the context of decision making is provided.

The aim of this paper is also to compare three algorithms which follow from the theoretical results presented here. These algorithms (their complexity) and theoretical results (assumptions on functions used in aggregation process) are compared in order to obtain the most useful practically result. The assumptions on  $F$  which are used to aggregate  $R_1, \dots, R_n$  are the minimal ones, i.e. we do not necessarily consider aggregation functions  $F$  but just functions  $F : [0, 1]^2 \rightarrow [0, 1]$ , which were recently called ‘fusion functions’ [6]. If it comes to complexity, it turned out that it is the same for each presented algorithm (for a given property). In the case of assumptions on fusion functions  $F$  the situation may be different what is analyzed in Section 7.1.

In Section 2, useful definitions are collected. In Section 3, motivation from real-life situations to consider such theoretical problem is presented. In Section 4, diverse dependencies and interpretation of  $\alpha$ -properties are discussed. In Section 5, graded properties: reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness and transitivity are examined one by one, in the context of their preservation in aggregation process. In Section 6, reciprocity property and other concepts and properties connected with decision making algorithms are mentioned. Finally, in Section 7 comparison of algorithms based on the theoretical studies presented in this paper are pro-

vided.

## 2. Preliminaries

Now we recall some definitions which will be helpful in our investigations.

**Definition 1** ([38]). *A fuzzy relation in  $X \neq \emptyset$  is a function  $R : X \times X \rightarrow [0, 1]$ . The family of all fuzzy relations in  $X$  is denoted by  $\mathcal{FR}(X)$ .*

With the use of  $n$ -argument functions  $F$  we aggregate given fuzzy relations  $R_1, \dots, R_n$  for a fixed  $n \in \mathbb{N}$ .

**Definition 2** ([25]). *Let  $F : [0, 1]^n \rightarrow [0, 1]$ ,  $R_1, \dots, R_n \in \mathcal{FR}(X)$ .  $R_F \in \mathcal{FR}(X)$ , where*

$$R_F(x, y) = F(R_1(x, y), \dots, R_n(x, y)), \quad x, y \in X,$$

*will be called an aggregated fuzzy relation. A function  $F$  preserves a property of fuzzy relations if for every relation  $R_1, \dots, R_n \in \mathcal{FR}(X)$  having this property,  $R_F$  also has this property.*

**Example 1.** *Projections  $P_k(t_1, \dots, t_n) = t_k$ ,  $k \in \{1, \dots, n\}$  preserve each property of fuzzy relations because for  $F = P_k$  we get  $R_F = R_k$ .*

**Definition 3** ([7]). *Let  $n \geq 2$ . A function  $F : [0, 1]^n \rightarrow [0, 1]$  is called an aggregation function, if it is increasing with respect to any variable, i.e. for any  $s_1, \dots, s_n, t_1, \dots, t_n \in [0, 1]$*

$$\left( \bigwedge_{1 \leq k \leq n} s_k \leq t_k \right) \Rightarrow F(s_1, \dots, s_n) \leq F(t_1, \dots, t_n) \quad (1)$$

and  $F(0, \dots, 0) = 0, F(1, \dots, 1) = 1$ .

**Definition 4** ([15]). *An operation  $C : [0, 1]^2 \rightarrow [0, 1]$  is called a fuzzy conjunction if it is increasing and*

$$C(1, 1) = 1, \quad C(0, 0) = C(0, 1) = C(1, 0) = 0.$$

*An operation  $D : [0, 1]^2 \rightarrow [0, 1]$  is called a fuzzy disjunction if it is increasing and*

$$D(0, 0) = 0, \quad D(1, 1) = D(0, 1) = D(1, 0) = 1.$$

Fuzzy conjunctions and disjunctions are examples of binary aggregation functions. Conversely, if a binary aggregation function has a zero element  $z = 0$  (as in the case of the geometric mean), then it is a fuzzy conjunction. Similarly, if a binary aggregation function has a zero element  $z = 1$ , then we get a fuzzy disjunction.

**Definition 5.** *A fuzzy conjunction which has a neutral element 1 is called a  $t$ -seminorm [20] (a semicopula [1], a conjunctor [9]). A fuzzy disjunction which has a neutral element 0 is called a  $t$ -semiconorm.*

**Corollary 1.** *If an operation  $B : [0, 1]^2 \rightarrow [0, 1]$  is increasing and has a neutral element 1 (neutral element 0), then it is a fuzzy conjunction fulfilling property  $B(x, y) \leq \min(x, y)$  (fuzzy disjunction fulfilling property  $B(x, y) \geq \max(x, y)$ ).*

Triangular norms and conorms are examples of conjunctions and disjunctions having neutral element 1 or 0, respectively.

**Definition 6** ([28]). *A triangular norm  $T : [0, 1]^2 \rightarrow [0, 1]$  (a triangular conorm  $S : [0, 1]^2 \rightarrow [0, 1]$ ) is an arbitrary associative, commutative, increasing in both variables function having a neutral element  $e = 1$  ( $e = 0$ ).*

Basic triangular norms and conorms are presented below.

**Example 2** ([28], p. 6). *For arbitrary  $s, t \in [0, 1]$  we have functions:*

- *lattice,  $T_M(s, t) = \min(s, t)$ ,  $S_M(s, t) = \max(s, t)$ ,*
- *Łukasiewicz,  $T_L(s, t) = \max(s + t - 1, 0)$ ,*

$$S_L(s, t) = \min(s + t, 1),$$

- *product,  $T_P(s, t) = st$ ,  $S_P(s, t) = s + t - st$ ,*

$$\bullet \text{ drastic, } T_D(s, t) = \begin{cases} 0, & s, t < 1 \\ s, & t = 1 \\ t, & s = 1 \end{cases},$$

$$S_D(s, t) = \begin{cases} 1, & s, t > 0 \\ s & t = 0 \\ t, & s = 0 \end{cases}.$$

Thanks to the associativity property triangular norms and conorms may be extended to  $n$ -argument functions. Special case of aggregation functions are the ones which are idempotent.

**Lemma 1** ([25], Proposition 5.1). *Every function  $F : [0, 1]^n \rightarrow [0, 1]$  increasing in each variable and idempotent*

$$\bigwedge_{t \in [0, 1]} F(t, \dots, t) = t \quad (2)$$

*fulfils for any  $t_1, \dots, t_n \in [0, 1]$*

$$\min(t_1, \dots, t_n) \leq F(t_1, \dots, t_n) \leq \max(t_1, \dots, t_n). \quad (3)$$

Here we present examples of functions which fulfil (3).

**Example 3.** *Let  $\varphi : [0, 1] \rightarrow [0, 1]$  be a continuous, strictly monotonic function. A quasi-linear mean (cf. [25], p. 112) is the function*

$$F(t_1, \dots, t_n) = \varphi^{-1} \left( \sum_{i=1}^n w_i \varphi(t_i) \right), \quad t_1, \dots, t_n \in [0, 1],$$

where  $\sum_{i=1}^n w_i = 1, w_i \in [0, 1]$ . Particularly, we obtain weighted arithmetic means

$$F(t_1, \dots, t_n) = \sum_{i=1}^n w_i t_i, \quad t_1, \dots, t_n \in [0, 1],$$

where  $\sum_{i=1}^n w_i = 1, w_i \in [0, 1]$ . An aggregation function

$$F(t_1, \dots, t_n) = p \max_{1 \leq k \leq n} t_k + (1 - p) \min_{1 \leq k \leq n} t_k \quad (4)$$

is idempotent, where  $p \in (0, 1)$  is a parameter.

There are some connections between functions. For example, we may consider relation of dominance of one function over another.

**Definition 7** (cf. [36], [34]). Let  $m, n \in \mathbb{N}$ . A function  $F : [0, 1]^m \rightarrow [0, 1]$  dominates a function  $G : [0, 1]^n \rightarrow [0, 1]$  ( $F \gg G$ ), if for arbitrary matrix  $[a_{ik}] = A \in [0, 1]^{m \times n}$  we have

$$F(G(a_{11}, \dots, a_{1n}), \dots, G(a_{m1}, \dots, a_{mn})) \geq G(F(a_{11}, \dots, a_{m1}), \dots, F(a_{1n}, \dots, a_{mn})). \quad (5)$$

**Lemma 2.** Let  $G : [0, 1]^n \rightarrow [0, 1]$  be increasing,  $m = 2$  (cf. (5)). Thus  $\min \gg G$  ([34], p. 16) and  $G \gg \max$  (cf. [11], Theorem 2), so for  $s_1, \dots, s_n, t_1, \dots, t_n \in [0, 1]$  we have respectively

$$\min(G(s_1, \dots, s_n), G(t_1, \dots, t_n)) \geq G(\min(s_1, t_1), \dots, \min(s_n, t_n)) \quad (6)$$

and

$$G(\max(s_1, t_1), \dots, \max(s_n, t_n)) \geq \max(G(s_1, \dots, s_n), G(t_1, \dots, t_n)). \quad (7)$$

**Theorem 1.** An increasing in each variable function  $F : [0, 1]^n \rightarrow [0, 1]$  dominates minimum ( $F \gg \min$ ) if and only if

$$F(t_1, \dots, t_n) = \min(f_1(t_1), \dots, f_n(t_n)), \quad t_1, \dots, t_n \in [0, 1], \quad (8)$$

where functions  $f_k : [0, 1] \rightarrow [0, 1]$  are increasing for  $k = 1, \dots, n$  (cf. [34], Proposition 5.1).

An increasing in each variable function  $F : [0, 1]^n \rightarrow [0, 1]$  is dominated by maximum ( $\max \gg F$ ) if and only if

$$F(t_1, \dots, t_n) = \max(f_1(t_1), \dots, f_n(t_n)), \quad t_1, \dots, t_n \in [0, 1], \quad (9)$$

where functions  $f_k : [0, 1] \rightarrow [0, 1]$  are increasing for  $k = 1, \dots, n$ .

**Example 4** (cf. [31]). Here are examples of functions fulfilling (8):

if  $f_k(t) = t, k = 1, \dots, n$ , then  $F = \min$ ,  
if for some  $k \in \{1, \dots, n\}$ ,  $f_k(t) = t, f_i(t) = 1$  for  $i \neq k$ , then  $F = P_k$ ,  
if  $f_k(t) = \max(1 - v_k, t), v_k \in [0, 1], k = 1, \dots, n, \max_{1 \leq k \leq n} v_k = 1$ , then  $F$  is the weighted minimum

$$F(t_1, \dots, t_n) = \min_{1 \leq k \leq n} \max(1 - v_k, t_k), \quad (10)$$

where  $t = (t_1, \dots, t_n) \in [0, 1]^n$ .

Here are examples of functions fulfilling (9):

if  $f_k(t) = t, k = 1, \dots, n$ , then  $F = \max$ ,  
if for some  $k \in \{1, \dots, n\}$ ,  $f_k(t) = t, f_i(t) = 0$  for  $i \neq k$ , then  $F = P_k$ ,  
if  $f_k(t) = \min(v_k, t), v_k \in [0, 1], k = 1, \dots, n, \max_{1 \leq k \leq n} v_k = 1$ , then  $F$  is the weighted maximum

$$F(t_1, \dots, t_n) = \max_{1 \leq k \leq n} \min(v_k, t_k), \quad (11)$$

where  $t = (t_1, \dots, t_n) \in [0, 1]^n$ .

**Lemma 3** (cf. [18]). If a function  $F : [0, 1]^n \rightarrow [0, 1]$  is increasing in each variable and has a neutral element  $e = 1$ , i.e.

$$\bigvee_{t \in [0,1]} \bigvee_{1 \leq k \leq n} F(1, \dots, 1, t, 1, \dots, 1) = t, \quad (12)$$

where  $t$  is at the  $k$ -th position, then  $F \leq \min$ .

If a function  $F : [0, 1]^n \rightarrow [0, 1]$  is increasing in each variable and has a neutral element  $e = 0$ , i.e.

$$\bigvee_{t \in [0,1]} \bigvee_{1 \leq k \leq n} F(0, \dots, 0, t, 0, \dots, 0) = t, \quad (13)$$

where  $t$  is at the  $k$ -th position, then  $F \geq \max$ .

Here are recalled definitions of concepts connected with fuzzy relations.

**Definition 8** (cf. [38]). Let  $R \in \mathcal{FR}(X)$ ,  $\alpha \in [0, 1]$ . The  $\alpha$ -cut of a fuzzy relation  $R$  is the relation

$$R_\alpha = \{(x, y) \in X \times X : R(x, y) \geq \alpha\}. \quad (14)$$

The strict  $\alpha$ -cut of a fuzzy relation  $R$  is the relation

$$R^\alpha = \{(x, y) \in X \times X : R(x, y) > \alpha\}. \quad (15)$$

**Definition 9** (cf. [38]). Let  $R, S \in \mathcal{FR}(X)$ . The composition of relations  $R$  and  $S$  is called the relation

$$(R \circ S)(x, z) = \sup_{y \in X} \min(R(x, y), S(y, z)), \quad (x, z) \in X \times X. \quad (16)$$

The power of a relation  $R$  is called the sequence  $R^1 = R$  and  $R^{n+1} = R^n \circ R$  for  $n \in \mathbb{N}$ .

**Remark 1.** If  $\text{card } X = n, X = \{x_1, \dots, x_n\}$ , then a relation  $R \in \mathcal{FR}(X)$  may be presented by a matrix  $R = [r_{ik}]$ , where  $r_{ik} = R(x_i, x_k), i, k = 1, \dots, n$ .

### 3. Motivation

In this section the idea of multicriteria (or similarly multiagent) decision making is recalled. Presented problem is related to considerations provided in this paper. Fuzzy relations in such setting represent the preferences.

Let  $\text{card } X = m, m \in \mathbb{N}, X = \{x_1, \dots, x_m\}$  be a set of alternatives. In multicriteria decision making a decision maker has to choose among the alternatives with respect to a set of criteria. Let  $K = \{k_1, \dots, k_n\}$  be the set of criteria on the base of which the alternatives are evaluated.  $R_1, \dots, R_n$  be fuzzy relations corresponding to each criterion represented by matrices, where  $R_k : X \times X \rightarrow [0, 1], k = 1, \dots, n, n \in \mathbb{N}, R_k(x_i, x_j) = r_{ij}^k, 1 \leq i, j \leq m$ . We assume that for example:  
 $r_{ij}^k$  – an intensity with which  $x_i$  is better than  $x_j$  under  $k \in K$ ,  
 $r_{ij}^k = 1$  – „ $x_i$  is absolutely better than  $x_j$  under criterion  $k$ ”,  
 $r_{ij}^k = 0$  – „ $x_j$  is absolutely better than  $x_i$  under criterion  $k$ ”,  
 $r_{ij}^k = 0.5$  – „ $x_i$  is equally good as  $x_j$  under criterion  $k$ ”,  
so it is natural that  $r_{ii}^k = 0.5$ .

Similarly, if we consider multiagent decision making problems, relations  $R_1, \dots, R_n$  represent the preferences of each agent and no criteria (certainly, we can combine these two situations, i.e. many criteria and many agents).

Relation  $R_F = F(R_1, \dots, R_n)$  is supposed to help the decision maker to make up his/her mind. Some functions  $F$  maybe more adequate for aggregation than the others since they may (or not) preserve the required properties of individual fuzzy relations  $R_1, \dots, R_n$ . According to some experimental works [40] weighted arithmetic mean and function (4) are the aggregation functions which occur the most often in the process of human decision making. Such properties, if they are fulfilled by fuzzy relations, may be a form of measure of consistency of choices or may provide the interpretation of choices. This is why preservation of these properties may be interested and required in aggregation process for multicriteria or multiagent decision making problems.

Application of similar considerations by a numerical example is presented in [32] where the choice or ranking problems of a set of alternatives evaluated by fuzzy preference relations using the aggregation functions are considered. It is shown how properties of the aggregated fuzzy relation  $R_F = F(R_1, \dots, R_n)$ , depending on the properties of the individual fuzzy relations  $R_1, \dots, R_n$ , help to solve the given problem. However, in that paper it is stressed also another problem, namely the sensitivity of the aggregation operators with respect to variations in their arguments. In that paper several weighted aggregation operators, i.e. operators which use the importance of criteria, given as weights, are considered.

In the presented multicriteria or multiagent decision making problems it is sometimes required that the given fuzzy relations representing the preferences are reciprocal, i.e. fuzzy relation  $R$  in  $X$  is reciprocal if  $R(x, y) + R(y, x) = 1$  for  $x, y \in X$ . However, if  $R$  is not reciprocal, there are methods to transform it to the reciprocal one [3].

#### 4. Graded Properties of Fuzzy Relations

Now, dependencies related to  $\alpha$ -properties in the context of aggregation process, between relations  $R_1, \dots, R_n$  on a set  $X$  and the aggregated fuzzy relation  $R_F = F(R_1, \dots, R_n)$  will be investigated. Moreover, some previous results will be recalled.

**Definition 10** ([10], p. 75, [18]). *Let  $\alpha \in [0, 1]$ . A relation  $R \in \mathcal{FR}(X)$  is:*

- $\alpha$ -reflexive, if  $\forall_{x \in X} R(x, x) \geq \alpha$ ,
- $\alpha$ -irreflexive, if  $\forall_{x \in X} R(x, x) \leq 1 - \alpha$ ,
- totally  $\alpha$ -connected, if  $\forall_{x, y \in X} \max(R(x, y), R(y, x)) \geq \alpha$ ,
- $\alpha$ -connected, if  $\forall_{x, y, x \neq y \in X} \max(R(x, y), R(y, x)) \geq \alpha$ ,
- $\alpha$ -asymmetric, if  $\forall_{x, y \in X} \min(R(x, y), R(y, x)) \leq 1 - \alpha$ ,

- $\alpha$ -antisymmetric, if  $\forall_{x, y, x \neq y \in X} \min(R(x, y), R(y, x)) \leq 1 - \alpha$ ,
- $\alpha$ -symmetric, if  $\forall_{x, y \in X} R(x, y) \geq 1 - \alpha \Rightarrow R(y, x) \geq R(x, y)$ ,
- $\alpha$ -transitive, if for all  $x, y, z \in X$   
 $\min(R(x, y), R(y, z)) \geq 1 - \alpha \Rightarrow R(x, z) \geq \min(R(x, y), R(y, z))$ .

Let us notice that conditions for  $\alpha$ -symmetry and  $\alpha$ -transitivity may be written in a more convenient way.

**Corollary 2.** *Let  $\alpha \in [0, 1]$ . A relation  $R \in \mathcal{FR}(X)$  is  $\alpha$ -symmetric if and only if*

$$\forall_{x, y \in X} R(x, y) \geq 1 - \alpha \Rightarrow R(y, x) = R(x, y). \quad (17)$$

**Corollary 3** (cf. [13], Theorem 10). *Let  $R \in \mathcal{FR}(X)$ ,  $\alpha \in [0, 1]$ . Relation  $R$  is  $\alpha$ -transitive if and only if*

$$R^2 \geq 1 - \alpha \Rightarrow R \geq R^2. \quad (18)$$

**Corollary 4.** *Let  $R \in \mathcal{FR}(X)$ ,  $\beta \in [0, 1]$ . If relation  $R$  is  $\beta$ - $P$ , then it is  $\alpha$ - $P$  for any  $\alpha \in [0, \beta]$ , where  $P$ : reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness, total connectedness, transitivity.*

*Proof.* Let  $\alpha \leq \beta$ . We use the fact, which is easy to see for each property  $\alpha$ - $P$ , where  $P$ : reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness, total connectedness, transitivity, that if  $R \in \mathcal{FR}(X)$  is  $\beta$ - $P$ , then it is  $\alpha$ - $P$ .  $\square$

If we have a reciprocal fuzzy relation  $R$  describing preferences, then the properties in Definition 10 may provide some practical information according to the preferences over the given set of alternatives. For example, the 0.5-asymmetry of a reciprocal fuzzy relation guarantees that at least one of the alternatives  $x_i$  or  $x_j$  is preferred to the other one with the fuzzy value lower than or equal to 0.5 (or these alternatives are indifferent), which means that if  $x_i$  is preferred to  $x_j$ , then it is not true that  $x_j$  is preferred to  $x_i$ . This interpretation of 0.5-asymmetry for a reciprocal fuzzy relation is analogous to the one of asymmetry for crisp relations (i.e., if element  $x_i$  is in relation with  $x_j$ , then it is not true that  $x_j$  is in relation with  $x_i$  [33]). Similarly, we can interpret the other properties. For  $\alpha$ -connectedness, the greater the value of  $\alpha$  (namely, the closer it is to the value 1), the choice of the alternative is more precise (confident/sure). For preference relations, if it comes to  $\alpha$ -reflexivity, practically only 0.5-reflexivity occurs and with the given definition, if  $R$  is 0.5-reflexive then it is automatically 0.5-irreflexive. Moreover, reciprocal preference relation is always totally 0.5-connected and 0.5-asymmetric. Since the fixed value of 0.5 on the diagonal may understate or inflate the value of  $\alpha$  for these properties, it makes sense to distinguish total connectedness and connectedness and similarly, asymmetry and antisymmetry.  $R \in \mathcal{FR}(X)$  has the highest value of  $\alpha$ -symmetry for preference relation  $R$  in the case when



all elements in the set of alternatives  $X$  are indifferent. In fact, in such situation relation  $R$  is symmetric (it is  $\alpha$ -symmetric for  $\alpha \in [0, 1]$ ). Moreover, we have the following statement: if  $R \in \mathcal{FR}(X)$  is reciprocal, then  $R$  is totally  $\alpha$ -connected ( $\alpha$ -connected) if and only if  $R$  is  $\alpha$ -asymmetric ( $\alpha$ -antisymmetric). That is not the case for  $R \in \mathcal{FR}(X)$  which is not reciprocal (cf. Example 5). In the sequel we will present the results in general setting of fuzzy relations, sometimes with the comments on reciprocal preference relations.

For practical reasons it is useful to find the greatest value of  $\alpha$  for which  $R \in \mathcal{FR}(X)$  is  $\alpha$ - $P$  for a given property  $P$ : reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness, total connectedness, transitivity. Applying definitions of the given properties and Corollary 4 one can find this value in the following way.

**Corollary 5.** Let  $R \in \mathcal{FR}(X)$ ,

$$\alpha_0 = 1 - \sup_{x,y \in X} \min(R(x,y), R(y,x)),$$

$$\beta_0 = 1 - \sup_{x \neq y} \min(R(x,y), R(y,x)),$$

$$\gamma_0 = \inf_{x,y \in X} \max(R(x,y), R(y,x)),$$

$$\delta_0 = \inf_{x \neq y} \max(R(x,y), R(y,x)),$$

$$\mu_0 = \inf_{x \in X} R(x,x),$$

$$\nu_0 = \inf_{x \in X} (1 - R(x,x)) = 1 - \sup_{x \in X} R(x,x).$$

Thus a relation  $R$  is:  $\alpha$ -asymmetric for  $\alpha \in [0, \alpha_0]$ ,  $\beta$ -antisymmetric for  $\beta \in [0, \beta_0]$ , totally  $\gamma$ -connected for  $\gamma \in [0, \gamma_0]$ ,  $\delta$ -connected for  $\delta \in [0, \delta_0]$ ,  $\mu$ -reflexive for  $\mu \in [0, \mu_0]$  and  $\nu$ -irreflexive for  $\nu \in [0, \nu_0]$ .

For symmetry and transitivity we have adequate half-closed intervals. Moreover, for checking the  $\alpha$ -transitivity of a fuzzy relation  $R$ , the composition of  $R$  by itself will be useful.

**Corollary 6.** Let  $R \in \mathcal{FR}(X)$ . Thus  $R$  is  $\alpha$ -symmetric for  $\alpha \in [0, 1]$  if  $R(x,y) = R(y,x)$  for all  $x,y \in X$  or  $R$  is  $\alpha$ -symmetric for  $\alpha \in [0, \alpha_0]$  if there exist  $x,y \in X$  such that  $R(x,y) \neq R(y,x)$ , where

$$\alpha_0 = 1 - \sup_{R(x,y) \neq R(y,x), x,y \in X} R(x,y).$$

$R$  is  $\beta$ -transitive for  $\beta \in [0, 1]$  if  $R^2(x,y) \leq R(x,y)$  for all  $x,y \in X$  or  $R$  is  $\beta$ -transitive for  $\beta \in [0, \beta_0]$  if there exist  $x,y \in X$  such that  $R(x,y) < R^2(x,y)$ , where

$$\beta_0 = 1 - \sup_{R(x,y) < R^2(x,y), x,y \in X} R^2(x,y).$$

**Example 5.** Let  $\text{card } X = 2$ ,  $R \in \mathcal{FR}(X)$ , where

$$R = \begin{bmatrix} 0.7 & 0.2 \\ 0.5 & 0.4 \end{bmatrix}.$$

The relation  $R$  is totally  $\alpha$ -connected and  $\alpha$ -reflexive for  $\alpha \in [0, 0.4]$  and  $\alpha$ -connected for  $\alpha \in [0, 0.5]$ . It is  $\alpha$ -asymmetric and  $\alpha$ -irreflexive for  $\alpha \in [0, 0.3]$  and

$\alpha$ -antisymmetric for  $\alpha \in [0, 0.8]$ .  $R$  is  $\alpha$ -symmetric for  $\alpha \in [0, 0.5]$  and  $\alpha$ -transitive for  $\alpha \in [0, 1]$  (it follows from the fact that  $R^2 = R$ ).

Let  $\text{card } X = 3$ . We consider  $R \in \mathcal{FR}(X)$  which is reciprocal, where

$$R = \begin{bmatrix} 0.5 & 0.8 & 0.3 \\ 0.2 & 0.5 & 0.4 \\ 0.7 & 0.6 & 0.5 \end{bmatrix}, \quad R^2 = \begin{bmatrix} 0.5 & 0.5 & 0.4 \\ 0.4 & 0.5 & 0.4 \\ 0.5 & 0.7 & 0.5 \end{bmatrix}.$$

The relation  $R$  is totally  $\alpha$ -connected and  $\alpha$ -reflexive for  $\alpha \in [0, 0.5]$  and  $\alpha$ -connected for  $\alpha \in [0, 0.6]$ . It is  $\alpha$ -asymmetric and  $\alpha$ -irreflexive for  $\alpha \in [0, 0.5]$  and  $\alpha$ -antisymmetric for  $\alpha \in [0, 0.6]$ .  $R$  is  $\alpha$ -symmetric for  $\alpha \in [0, 0.2)$  and  $\alpha$ -transitive for  $\alpha \in [0, 0.3)$ .

**Remark 2.** The presented  $\alpha$ -properties (graded properties) for  $\alpha = 1$  become the basic properties of fuzzy relations [39]. Graded properties are „fuzzy versions“ of properties introduced by Zadeh. It means that, if a fuzzy relation, e.g. is not reflexive, it may be reflexive to some grade  $\alpha$ , where  $\alpha \in [0, 1]$ .

**Remark 3.** Taking into account  $\alpha = 0$ , each fuzzy relation is 0-reflexive, 0-irreflexive, 0-asymmetric, 0-antisymmetric, 0-connected and totally 0-connected. However, it is not true for graded symmetry and transitivity. If in Corollary 6,  $\alpha_0 = 0$  (or similarly  $\beta_0 = 0$ ), then  $R$  is not  $\alpha$ -symmetric for any  $\alpha \in [0, 1]$  ( $R$  is not  $\beta$ -transitive for any  $\beta \in [0, 1]$ ).

**Example 6.** Let  $\text{card } X = 3$ , relations  $R, S \in \mathcal{FR}(X)$  be presented by matrices:

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The relation  $R$  is not 0-transitive because  $\min(r_{13}, r_{31}) = 1$  but  $0 = r_{11} < \min(r_{13}, r_{31}) = 1$ . The relation  $S$  is not 0-symmetric because  $s_{31} = 1$  and  $0 = s_{13} < s_{31} = 1$ .

Notions of  $\alpha$ -properties have their connection with cuts and strict cuts of a fuzzy relation.

**Theorem 2** (cf. [17]). Let  $\alpha \in [0, 1]$ ,  $R \in \mathcal{FR}(X)$ . A fuzzy relation  $R$  is totally  $\alpha$ -connected ( $\alpha$ -connected,  $\alpha$ -reflexive) if and only if relation  $R_\alpha$  is totally connected (connected, reflexive). A fuzzy relation  $R$  is  $\alpha$ -asymmetric ( $\alpha$ -antisymmetric,  $\alpha$ -irreflexive) if and only if relation  $R^{1-\alpha}$  is asymmetric (antisymmetric, irreflexive). If a fuzzy relation  $R$  is  $\alpha$ -transitive, then relation  $R_{1-\alpha}$  is transitive. If a fuzzy relation  $R$  is  $\alpha$ -symmetric, then relation  $R_{1-\alpha}$  is symmetric.

Similar characterizations for other properties for fuzzy relations one may find in [12] (Theorem 1). The conditions for  $\alpha$ -symmetry and  $\alpha$ -transitivity are only the sufficient ones.

**Example 7** (cf. [17]). Let  $\text{card } X = 2$ ,  $R \in \mathcal{FR}(X)$ ,

$$R = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.4 \end{bmatrix}.$$

The cuts  $R_\beta$  are symmetric for  $\beta \in [0, 0.5] \cup (0.7, 1]$ , so the cuts  $R_{1-\alpha}$  have this property for  $\alpha \geq 0.5$  and  $\alpha < 0.3$ . Relation  $R$  is  $\alpha$ -symmetric for  $\alpha \in [0, 0.3]$ , as a result for  $\alpha = 0.5$ , the cut  $R_{0.5}$  is symmetric, while  $R$  is not 0.5-symmetric.

Let  $R \in \mathcal{FR}(X)$ , card  $X = 3$ ,

$$R = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.8 & 0.9 & 0 \\ 0.6 & 0.9 & 0.8 \end{bmatrix}, \quad S = R^2 = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.8 & 0.9 & 0 \\ 0.8 & 0.9 & 0.8 \end{bmatrix}.$$

The cuts  $R_\beta$  are transitive for  $\beta \in [0, 0.6] \cup (0.8, 1]$ , so the cuts  $R_{1-\alpha}$  have this property for  $\alpha \in [0, 0.2] \cup [0.4, 1]$ . Since  $0.8 = s_{32} \geq 1 - \alpha$  for  $\alpha \in [0.4, 1]$  and  $s_{32} = 0.8 > 0.6 = r_{32}$ , relation  $R$  is not  $\alpha$ -transitive for  $\alpha \in [0.4, 1]$  (it is  $\alpha$ -transitive for  $\alpha \in [0, 0.2]$ , see Corollary 6).

Other results describing graded properties one can find in [10] (p. 78-79).

### 5. Aggregation of Fuzzy Relations

In this section we will present  $\alpha$ -properties of fuzzy relations and diverse approaches of aggregating such relations. There will be presented the following type of theorems for aggregated fuzzy relation  $R_F$ :

- aggregation of  $R_1, \dots, R_n$  all having the same grade of a given  $\alpha$ -property to obtain  $R_F$  with the same grade  $\alpha$ ,
- aggregation of  $R_1, \dots, R_n$  with possible diverse grades  $\alpha_1, \dots, \alpha_n$  of a given graded property to obtain  $R_F$  with the suitable grade  $\alpha$ ,
- starting from  $R_F$  having some grade  $\alpha$  and checking whether  $R_1, \dots, R_n$  all have the same grade  $\alpha$  of a given graded property.

#### 5.1. Reflexivity

Graded reflexivity was considered by many authors, e.g. [8], [10].

**Theorem 3** ([16]). Let  $\alpha \in [0, 1]$ .  $F : [0, 1]^n \rightarrow [0, 1]$  preserves  $\alpha$ -reflexivity of fuzzy relations, if and only if

$$F|_{[\alpha, 1]^n} \geq \alpha.$$

**Theorem 4** ([16]).  $F : [0, 1]^n \rightarrow [0, 1]$  preserves  $\alpha$ -reflexivity of fuzzy relations for arbitrary  $\alpha \in [0, 1]$  if and only if  $F \geq \min$ .

By Lemma 1 we know that every increasing and idempotent function preserves  $\alpha$ -reflexivity of fuzzy relations for arbitrary  $\alpha \in [0, 1]$ . In particular, we get

**Corollary 7.** Quasi-linear means preserve  $\alpha$ -reflexivity of fuzzy relations for any  $\alpha \in [0, 1]$ .

**Theorem 5.** Let  $\alpha_1, \dots, \alpha_n \in [0, 1]$ , a function  $F : [0, 1]^n \rightarrow [0, 1]$  be increasing in each variable. If relations  $R_i \in \mathcal{FR}(X)$  are  $\alpha_i$ -reflexive for  $i = 1, \dots, n$ , then relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -reflexive for  $\alpha = F(\alpha_1, \dots, \alpha_n)$ .

*Proof.* Let  $\alpha_1, \dots, \alpha_n \in [0, 1]$ , a function  $F : [0, 1]^n \rightarrow [0, 1]$  be increasing in each variable,  $R_i \in \mathcal{FR}(X)$  be  $\alpha_i$ -reflexive for  $i = 1, \dots, n, x \in X$ . Then

$$R(x, x) = F(R_1(x, x), \dots, R_n(x, x)) \geq F(\alpha_1, \dots, \alpha_n),$$

so relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -reflexive for  $\alpha = F(\alpha_1, \dots, \alpha_n)$ .  $\square$

Each aggregation function is increasing, so we get

**Corollary 8.** Let  $\alpha_1, \dots, \alpha_n \in [0, 1]$ ,  $F : [0, 1]^n \rightarrow [0, 1]$  be an aggregation function. If relations  $R_i \in \mathcal{FR}(X)$  are  $\alpha_i$ -reflexive for  $i = 1, \dots, n$ , then relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -reflexive for  $\alpha = F(\alpha_1, \dots, \alpha_n)$ .

**Theorem 6.** Let  $\alpha \in [0, 1]$  and  $F \leq \min$ . If a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -reflexive, then all relations  $R_1, \dots, R_n$  are  $\alpha$ -reflexive.

*Proof.* Let  $\alpha \in [0, 1]$ ,  $F \leq \min$ ,  $R_F = F(R_1, \dots, R_n)$  be  $\alpha$ -reflexive,  $x \in X, k \in \{1, \dots, n\}$ . Then

$$R_k(x, x) \geq \min_{1 \leq i \leq n} R_i(x, x) \geq$$

$$F(R_1(x, x), \dots, R_n(x, x)) \geq \alpha.$$

As a result relation  $R_k$  is  $\alpha$ -reflexive.  $\square$

In virtue of Lemma 3 we get

**Corollary 9.** Let  $\alpha \in [0, 1]$ ,  $F$  be a  $t$ -seminorm or a  $t$ -norm. If a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -reflexive, then all relations  $R_1, \dots, R_n$  are also  $\alpha$ -reflexive.

The next example shows that the condition presented in Theorem 6 is only sufficient.

**Example 8.** Let card  $X = 2$ . We consider fuzzy relations with matrices:

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$W_1 = \max(R, S) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$W_2 = \frac{R + S}{2} = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}.$$

Relation  $W_1$  is  $\alpha$ -reflexive for  $\alpha \in [0, 1]$ ,  $W_2$  for  $\alpha \in [0, 0.5]$ , but relations  $R, S$  do not have this property for any  $\alpha \in (0, 1]$ .

#### 5.2. Irreflexivity

For irreflexivity, generally we get dual results to reflexivity.

**Theorem 7** ([16]). Let  $\alpha \in [0, 1]$ . A function  $F : [0, 1]^n \rightarrow [0, 1]$  preserves  $\alpha$ -irreflexivity of fuzzy relations if and only if

$$F|_{[0, 1-\alpha]^n} \leq 1 - \alpha.$$

**Theorem 8** ([16]). A function  $F : [0, 1]^n \rightarrow [0, 1]$  preserves  $\alpha$ -irreflexivity of fuzzy relations for arbitrary  $\alpha \in [0, 1]$  if and only if  $F \leq \max$ .

**Corollary 10.** *Quasi-linear means preserve  $\alpha$ -irreflexivity of fuzzy relations for arbitrary  $\alpha \in [0, 1]$ .*

**Definition 11** (cf. [7]). *A function  $F : [0, 1]^n \rightarrow [0, 1]$  is super additive, if for all  $i = 1, \dots, n$  and all  $x_i, y_i, x_i + y_i \in [0, 1]$*

$$F(x_1 + y_1, \dots, x_n + y_n) \geq F(x_1, \dots, x_n) + F(y_1, \dots, y_n). \tag{19}$$

**Example 9.** *Weighted arithmetic means and minimum are super additive functions.*

**Theorem 9.** *Let  $\alpha_1, \dots, \alpha_n \in [0, 1]$ ,  $F : [0, 1]^n \rightarrow [0, 1]$  be a super additive aggregation function. If relations  $R_i \in \mathcal{FR}(X)$  are  $\alpha_i$ -irreflexive for  $i = 1, \dots, n$ , then relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -irreflexive for  $\alpha = F(\alpha_1, \dots, \alpha_n)$ .*

*Proof.* Let  $\alpha_1, \dots, \alpha_n \in [0, 1]$ ,  $F : [0, 1]^n \rightarrow [0, 1]$  be a super additive aggregation function,  $R_i \in \mathcal{FR}(X)$  be  $\alpha_i$ -irreflexive for  $i = 1, \dots, n$ ,  $x \in X$ . Then  $R_i(x, x) + \alpha_i \leq 1$ , so

$$F(R_1(x, x), \dots, R_n(x, x)) + F(\alpha_1, \dots, \alpha_n) \leq F(R_1(x, x) + \alpha_1, \dots, R_n(x, x) + \alpha_n) \leq F(1, \dots, 1) = 1.$$

As a result

$$F(R_1(x, x), \dots, R_n(x, x)) \leq 1 - F(\alpha_1, \dots, \alpha_n),$$

so  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -irreflexive for  $\alpha = F(\alpha_1, \dots, \alpha_n)$ .  $\square$

**Corollary 11.** *Let  $\alpha_1, \dots, \alpha_n \in [0, 1]$ . If relations  $R_i \in \mathcal{FR}(X)$  are  $\alpha_i$ -irreflexive for  $i = 1, \dots, n$ , then relation*

$$R = \sum_{i=1}^n w_i R_i \text{ is } \alpha\text{-irreflexive, where } \sum_{i=1}^n w_i = 1, w_i \in [0, 1] \text{ and } \alpha = \sum_{i=1}^n w_i \alpha_i.$$

Analogously to reflexivity we obtain the following result.

**Theorem 10.** *Let  $\alpha \in [0, 1]$  and  $F \geq \max$ . If a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -irreflexive, then all relations  $R_1, \dots, R_n$  are also  $\alpha$ -irreflexive.*

In virtue of Lemma 3 we get

**Corollary 12.** *Let  $\alpha \in [0, 1]$ ,  $F$  be a  $t$ -conorm or a  $t$ -semiconorm. If a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -irreflexive, then all relations  $R_1, \dots, R_n$  are also  $\alpha$ -irreflexive.*

The next example shows that the condition given in Theorem 10 is only sufficient.

**Example 10.** *Let card  $X = 2$ . We consider fuzzy relations with matrices:*

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$W_1 = \min(R, S) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$W_2 = \frac{R + S}{2} = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}.$$

*Relation  $W_1$  is  $\alpha$ -irreflexive for  $\alpha \in [0, 1]$ ,  $W_2$  for  $\alpha \in [0, 0.5]$ , but relations  $R, S$  do not have this property for any  $\alpha \in (0, 1]$ .*

### 5.3. Connectedness

Here graded connectedness and total connectedness will be examined. The total 0.5-connectedness was regarded in [32] (p. 619). In that paper this property is called weak comparability. It was shown there that maximum preserves the total 0.5-connectedness ([32], Table 1).

**Theorem 11** ([16]). *Let  $\alpha \in [0, 1]$ , card  $X \geq 2$ . A function  $F : [0, 1]^n \rightarrow [0, 1]$  preserves total  $\alpha$ -connectedness ( $\alpha$ -connectedness) of fuzzy relations, if and only if for any  $s, t \in [0, 1]^n$*

$$\left( \bigvee_{1 \leq k \leq n} \max(s_k, t_k) \geq \alpha \right) \Rightarrow \max(F(s), F(t)) \geq \alpha.$$

**Theorem 12** ([16]). *Let card  $X \geq 2$ . A function  $F : [0, 1]^n \rightarrow [0, 1]$  preserves total  $\alpha$ -connectedness ( $\alpha$ -connectedness) of fuzzy relations for arbitrary  $\alpha \in [0, 1]$ , if and only if*

$$\bigvee_{s, t \in [0, 1]^n} \max(F(s), F(t)) \geq \min_{1 \leq k \leq n} \max(s_k, t_k).$$

**Corollary 13.** *Maximum and the weighted maximum preserve total  $\alpha$ -connectedness ( $\alpha$ -connectedness) of fuzzy relations for arbitrary  $\alpha \in [0, 1]$ .*

**Theorem 13.** *Let  $\alpha_1, \dots, \alpha_n \in [0, 1]$ , a function  $F : [0, 1]^n \rightarrow [0, 1]$  be increasing in each variable and  $\max \gg F$ . If relations  $R_i \in \mathcal{FR}(X)$  are totally  $\alpha_i$ -connected ( $\alpha_i$ -connected) for  $i = 1, \dots, n$ , then relation  $R_F = F(R_1, \dots, R_n)$  is totally  $\alpha$ -connected ( $\alpha$ -connected) for  $\alpha = F(\alpha_1, \dots, \alpha_n)$ .*

*Proof.* Let  $\alpha_1, \dots, \alpha_n \in [0, 1]$ , a function  $F : [0, 1]^n \rightarrow [0, 1]$  be increasing in each variable,  $\max \gg F$  and  $R_i \in \mathcal{FR}(X)$  be  $\alpha_i$ -connected for  $i = 1, \dots, n$ ,  $x, y \in X, x \neq y$ . Then by Lemma 2 and by the fact that  $\max \gg F$  we obtain

$$\begin{aligned} \max(R(x, y), R(y, x)) &= \\ \max(F(R_1(x, y), \dots, R_n(x, y)), & \\ F(R_1(y, x), \dots, R_n(y, x))) &\geq \\ F(\max(R_1(x, y), R_n(x, y)), \dots, & \\ \max(R_n(x, y), R_n(y, x))) &\geq \\ F(\alpha_1, \dots, \alpha_n) &= \alpha. \end{aligned}$$

It means that a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -connected for  $\alpha = F(\alpha_1, \dots, \alpha_n)$ . Proof for total  $\alpha$ -connectedness is analogous.  $\square$

We can also compute the value of  $\alpha$  for which a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -connected (totally  $\alpha$ -connected) for concrete functions  $F$  in another way than it is presented in Theorem 13. It is shown in the following example.

**Example 11.** *Let  $\alpha_1, \dots, \alpha_n \in [0, 1]$ . If relations  $R_i \in \mathcal{FR}(X)$  are  $\alpha_i$ -connected (totally  $\alpha_i$ -connected) for  $i = 1, \dots, n$ , then relation  $R \in \mathcal{FR}(X)$  is  $\alpha$ -connected (totally  $\alpha$ -connected), where*

$$R = \frac{1}{n} \sum_{i=1}^n R_i, \quad \alpha = \frac{1}{n} \max_{1 \leq i \leq n} \alpha_i.$$

Note that the arithmetic mean is not dominated by maximum.

**Theorem 14.** Let  $\alpha \in [0, 1]$  and  $F \leq \min$ . If a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is totally  $\alpha$ -connected ( $\alpha$ -connected), then all fuzzy relations  $R_1, \dots, R_n$  are totally  $\alpha$ -connected ( $\alpha$ -connected).

*Proof.* Let  $\alpha \in [0, 1]$ ,  $F \leq \min$  and a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  be  $\alpha$ -connected,  $x, y \in X$ ,  $x \neq y$ ,  $k \in \{1, \dots, n\}$ . As a result we have  $\max(R(x, y), R(y, x)) \geq \alpha$ , so  $F(R_1(x, y), \dots, R_n(x, y)) = R(x, y) \geq \alpha$  or  $F(R_1(y, x), \dots, R_n(y, x)) = R(y, x) \geq \alpha$ . Let us consider the first case. Since  $F \leq \min$ , we get

$$R_k(x, y) \geq \min_{1 \leq i \leq n} R_i(x, y) \geq$$

$$F(R_1(x, y), \dots, R_n(x, y)) \geq \alpha.$$

It means that  $\max(R_k(x, y), R_k(y, x)) \geq \alpha$ . Similarly we may consider the second case, i.e.  $R(y, x) \geq \alpha$ . Thus relations  $R_i$  are  $\alpha$ -connected for  $i \in \{1, \dots, n\}$ . The proof for total  $\alpha$ -connectedness is analogous.  $\square$

By Lemma 3 we get

**Corollary 14.** Let  $\alpha \in [0, 1]$ ,  $F$  be a  $t$ -norm or a  $t$ -seminorm. If a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is totally  $\alpha$ -connected ( $\alpha$ -connected), then all fuzzy relations  $R_1, \dots, R_n$  are totally  $\alpha$ -connected ( $\alpha$ -connected).

**Example 12.** The condition given in Theorem 14 is only sufficient. For total  $\alpha$ -connectedness it is enough to consider relations from Example 8. Relation  $W_1$  is totally  $\alpha$ -connected for  $\alpha \in [0, 1]$ ,  $W_2$  for  $\alpha \in [0, 0.5]$ , but relations  $R, S$  do not have this property for any  $\alpha \in (0, 1)$ . For  $\alpha$ -connectedness let us take  $R = [r_{ij}]$ , with  $r_{ij} = 1$  and  $S = [s_{ij}]$ , with  $s_{ij} = 0$  for  $i, j = 1, \dots, n$ . Then relation  $W = \max(R, S) = R$  and  $R, W$  are  $\alpha$ -connected for  $\alpha \in [0, 1]$ , while  $S$  is not  $\alpha$ -connected for any  $\alpha \in (0, 1)$ .

### 5.4. Asymmetry

Now graded asymmetry and antisymmetry will be discussed. The obtained results are dual to the ones obtained for total  $\alpha$ -connectedness and  $\alpha$ -connectedness, respectively. It is worth mentioning that in [32] (p. 619) the 0.5-asymmetry was considered. However, in that paper this property is called weak asymmetry. It was shown there that minimum preserves the 0.5-asymmetry ([32], Table 1).

**Theorem 15** ([16]). Let  $\alpha \in [0, 1]$ ,  $\text{card } X \geq 2$ . A function  $F : [0, 1]^n \rightarrow [0, 1]$  preserves  $\alpha$ -asymmetry ( $\alpha$ -antisymmetry) of fuzzy relations, if and only if for any  $s, t \in [0, 1]^n$

$$(\bigvee_{1 \leq k \leq n} \min(s_k, t_k) \leq 1 - \alpha) \Rightarrow \min(F(s), F(t)) \leq 1 - \alpha.$$

**Theorem 16** ([16]). Let  $\text{card } X \geq 2$ . A function  $F : [0, 1]^n \rightarrow [0, 1]$  preserves  $\alpha$ -asymmetry ( $\alpha$ -antisymmetry) of fuzzy relations for arbitrary  $\alpha \in [0, 1]$ , if and only if

$$\bigvee_{s, t \in [0, 1]^n} \min(F(s), F(t)) \leq \max_{1 \leq k \leq n} \min(s_k, t_k).$$

**Corollary 15.** The minimum and the weighted minimum (11) preserve  $\alpha$ -asymmetry ( $\alpha$ -antisymmetry) of fuzzy relations for arbitrary  $\alpha \in [0, 1]$ .

Dually to graded connectedness properties, by Lemma 2, similarly to the proof of Theorem 9 we may prove

**Theorem 17.** Let  $\alpha_1, \dots, \alpha_n \in [0, 1]$ , a function  $F : [0, 1]^n \rightarrow [0, 1]$  be a super additive increasing in each variable function and  $F \gg \min$ . If relations  $R_i \in \mathcal{FR}(X)$  are totally  $\alpha_i$ -asymmetric ( $\alpha_i$ -antisymmetric) for  $i = 1, \dots, n$ , then relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -asymmetric ( $\alpha$ -antisymmetric) for  $\alpha = F(\alpha_1, \dots, \alpha_n)$ .

In Theorem 1 we have the characterization of increasing functions which dominate minimum. Appropriate examples are presented in Example 4 and among them minimum is a super additive function (because, by Lemma 2, it dominates any increasing function which coincides with the inequality (19)).

We can also compute the value of  $\alpha$  for which a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -asymmetric ( $\alpha$ -antisymmetric) for concrete functions  $F$  in another way than it is presented in Theorem 17. It is shown in the following example.

**Example 13.** Let  $\alpha_1, \dots, \alpha_n \in [0, 1]$ . If relations  $R_i \in \mathcal{FR}(X)$  are  $\alpha_i$ -asymmetric ( $\alpha_i$ -antisymmetric) for  $i = 1, \dots, n$ , then relation  $R \in \mathcal{FR}(X)$  is  $\alpha$ -asymmetric ( $\alpha$ -antisymmetric), where

$$R = \frac{1}{n} \sum_{i=1}^n R_i, \quad \alpha = \frac{1}{n} \min_{1 \leq i \leq n} \alpha_i.$$

Note that the arithmetic mean does not dominate minimum.

Dually to Theorem 14 we may prove

**Theorem 18.** Let  $\alpha \in [0, 1]$  and  $F \geq \max$ . If a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -asymmetric ( $\alpha$ -antisymmetric), then also all relations  $R_1, \dots, R_n$  are  $\alpha$ -asymmetric ( $\alpha$ -antisymmetric).

By Lemma 3 we obtain

**Corollary 16.** Let  $\alpha \in [0, 1]$ ,  $F$  be a  $t$ -conorm or a  $t$ -seminorm. If a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -asymmetric ( $\alpha$ -antisymmetric), then all relations  $R_1, \dots, R_n$  are  $\alpha$ -asymmetric ( $\alpha$ -antisymmetric).

**Example 14.** The condition given in Theorem 18 is only sufficient. For  $\alpha$ -asymmetry it is enough to consider relations from Example 10. The relation  $W_1$  is  $\alpha$ -asymmetric for  $\alpha \in [0, 1]$ ,  $W_2$  for  $\alpha \in [0, 0.5]$ , but relations  $R, S$  do not have this property for any  $\alpha \in (0, 1)$ . For  $\alpha$ -antisymmetry let us take  $R = [r_{ij}]$ , with  $r_{ij} = 1$  and  $S = [s_{ij}]$ , with  $s_{ij} = 0$  for  $i, j = 1, \dots, n$ . Then relation  $W = \min(R, S) = S$  and  $S, W$  are  $\alpha$ -antisymmetric for  $\alpha \in [0, 1]$ , while  $R$  is not  $\alpha$ -antisymmetric for any  $\alpha \in (0, 1)$ .



### 5.5. Symmetry

Now graded symmetry will be discussed.

**Theorem 19** ([18]). *Let  $\alpha \in [0, 1]$ . If a function  $F : [0, 1]^n \rightarrow [0, 1]$  fulfils*

$$F|_{[0,1]^n \setminus [1-\alpha,1]^n} < 1 - \alpha,$$

*then it preserves  $\alpha$ -symmetry of relations  $R_1, \dots, R_n \in \mathcal{FR}(X)$ .*

**Theorem 20** ([18]). *If a function  $F : [0, 1]^n \rightarrow [0, 1]$  fulfils condition  $F \leq \min$ , then it preserves  $\alpha$ -symmetry of fuzzy relations for arbitrary  $\alpha \in [0, 1]$ .*

**Corollary 17.** *Any triangular norm or a  $t$ -seminorm preserves  $\alpha$ -symmetry of fuzzy relations for arbitrary  $\alpha \in [0, 1]$ .*

**Example 15.** *Since any projection  $P_k$ ,  $k \in \{1, \dots, n\}$ , preserves the  $\alpha$ -symmetry for each  $\alpha \in [0, 1]$  but it is not true that  $P_k \leq \min$ , then Theorem 20 gives only a sufficient condition for preservation of the  $\alpha$ -symmetry for any  $\alpha \in [0, 1]$ .*

**Theorem 21.** *Let  $\alpha_1, \dots, \alpha_n \in [0, 1]$ ,  $F \leq \min$ . If relations  $R_i \in \mathcal{FR}(X)$  are  $\alpha_i$ -symmetric for  $i = 1, \dots, n$ , then relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -symmetric for  $\alpha = F(\alpha_1, \dots, \alpha_n)$ .*

*Proof.* Let relations  $R_i$  be  $\alpha_i$ -symmetric for  $i = 1, \dots, n$  and  $x, y \in X$ . If  $R(x, y) = F(R_1(x, y), \dots, R_n(x, y)) \geq 1 - \alpha$  and  $F \leq \min$ , then for  $k = 1, \dots, n$

$$R_k(x, y) \geq \min(R_1(x, y), \dots, R_n(x, y)) \geq 1 - \alpha = 1 - F(\alpha_1, \dots, \alpha_n).$$

Moreover, for  $k = 1, \dots, n$

$$1 - F(\alpha_1, \dots, \alpha_n) \geq 1 - \min(\alpha_1, \dots, \alpha_n) \geq 1 - \alpha_k.$$

As a result  $R_k(x, y) \geq 1 - \alpha_k$  for  $k = 1, \dots, n$ . It means that  $R_k(x, y) = R_k(y, x)$  for  $k = 1, \dots, n$ , so  $R(x, y) = R(y, x)$  and  $R$  is  $\alpha$ -symmetric for  $\alpha = F(\alpha_1, \dots, \alpha_n)$ .  $\square$

If it comes to the „converse problem” for  $\alpha$ -symmetry we have several counter-examples. Observe that diverse functions were applied for aggregation of fuzzy relations, namely greater (smaller) than or equal to minimum (maximum).

**Example 16.** *Let  $\text{card } X = 2$ . We consider fuzzy relations with matrices:*

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$W_1 = \min(R, S) = R \cdot S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$W_2 = \max(R, S) = R + S - R \cdot S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$W_3 = \frac{R + S}{2} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}.$$

*Relations  $W_1, W_2, W_3$  are  $\alpha$ -symmetric for  $\alpha \in [0, 1]$ , but relations  $R, S$  do not have this property for any  $\alpha \in [0, 1]$ .*

**Remark 4.** *Let  $\alpha \in [0, 1]$ . If we would assume that  $F$  is idempotent, increasing and injective with respect to all arguments, then if a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -symmetric, then also all relations  $R_1, \dots, R_n$  are  $\alpha$ -symmetric. However, idempotency and injectiveness with respect to all arguments makes a contraposition (if  $F(x, x) = x$ , then for the remaining arguments there are no values). Moreover, injectiveness with respect to all arguments, as a property itself, is not so easy to be fulfilled (arithmetic mean, minimum, maximum, geometric mean, uninorms – including  $t$ -norms and  $t$ -conorms, are not injective with respect to all arguments). Assuming injectiveness with respect to a fixed variable, i.e.  $F(x_1, y) = F(x_2, y) \Rightarrow x_1 = x_2$  for all  $y \in [0, 1]$ , in general ( $F$  should be without a zero element) is not contradictory with idempotency of  $F$ , but this assumption is not enough to obtain the required result which is shown by the counter-example above (relations  $R, S, W_3$  in Example 16, where the arithmetic mean is idempotent and injective with a fixed variable).*

### 5.6. Transitivity

In [31] a special case of the graded transitivity is considered. Namely, this is the 0.5-transitivity (there this property is called moderate transitivity). However, the problem of preservation of this property during aggregation process is not discussed. The property of the 0.5-transitivity is also known as one of the types of a stochastic transitivity (e.g. [19]).

**Theorem 22** ([18]). *Let  $\alpha \in [0, 1]$ . If an increasing function  $F : [0, 1]^n \rightarrow [0, 1]$  fulfils*

$$F|_{[0,1]^n \setminus [1-\alpha,1]^n} < 1 - \alpha,$$

*and  $F \gg \min$ , then it preserves  $\alpha$ -transitivity of fuzzy relations.*

**Example 17** ([18]). *Let  $a \in (0, 1)$  and  $F : [0, 1]^2 \rightarrow [0, 1]$  be of the form*

$$F(s, t) = \begin{cases} 0, & (s, t) \in [0, a) \times [0, a) \\ \min(s, t), & \text{otherwise} \end{cases}$$

*$F$  is a  $t$ -norm and  $F|_{[0,1]^2 \setminus [1-\alpha,1]^2} < 1 - \alpha$  but it does not dominate minimum. However, the function  $F$  preserves the  $\alpha$ -transitivity for each  $\alpha \in [0, 1)$  and  $\alpha \leq 1 - a$ . As a result conditions for preservation of the  $\alpha$ -transitivity stated in Theorem 22 are only sufficient.*

**Theorem 23** ([18]). *If a function  $F : [0, 1]^n \rightarrow [0, 1]$  is increasing in each variable, fulfils  $F \gg \min$  and  $F \leq \min$ , then it preserves  $\alpha$ -transitivity of fuzzy relations for any  $\alpha \in [0, 1]$ .*

**Corollary 18.** *Minimum and the aggregation function*

$$A_w(t_1, \dots, t_n) = \begin{cases} 1, & (t_1, \dots, t_n) = (1, \dots, 1) \\ 0, & \text{otherwise} \end{cases}$$

*preserve the  $\alpha$ -transitivity of fuzzy relations for any  $\alpha \in [0, 1]$  (because both functions fulfil assumptions of Theorem 23).*

**Theorem 24.** Let  $\alpha_1, \dots, \alpha_n \in [0, 1]$ ,  $F \leq \min$ ,  $F \gg \min$  and a function  $F$  be increasing. If relations  $R_i \in \mathcal{FR}(X)$  are  $\alpha_i$ -transitive for  $i = 1, \dots, n$ , then relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -transitive for  $\alpha = F(\alpha_1, \dots, \alpha_n)$ .

*Proof.* Let relations  $R_i$  be  $\alpha_i$ -transitive for  $i = 1, \dots, n$  and  $x, y, z \in X$ . If

$$\min(R(x, y), R(y, z)) =$$

$$\min(F(R_1(x, y), \dots, R_n(x, y)), F(R_1(y, z), \dots, R_n(y, z))) \geq 1 - \alpha$$

and  $F \leq \min$ , then by the monotonicity of minimum we get

$$\begin{aligned} & \min(R_k(x, y), R_k(y, z)) \geq \\ & \min(\min(R_1(x, y), \dots, R_n(x, y)), \\ & \min(R_1(y, z), \dots, R_n(y, z))) \geq \\ & 1 - \alpha = 1 - F(\alpha_1, \dots, \alpha_n) \end{aligned}$$

for  $k = 1, \dots, n$ . Moreover, for  $k = 1, \dots, n$

$$1 - F(\alpha_1, \dots, \alpha_n) \geq 1 - \min(\alpha_1, \dots, \alpha_n) \geq 1 - \alpha_k.$$

As a result  $\min(R_k(x, y), R_k(y, z)) \geq 1 - \alpha_k$  for  $k = 1, \dots, n$ . By assumptions it means that  $\min(R_k(x, y), R_k(y, z)) \leq R_k(x, z)$  for  $k = 1, \dots, n$ . Since  $F \gg \min$  and  $F$  is increasing, one obtains

$$\min(R(x, y), R(y, z)) =$$

$$\begin{aligned} & \min(F(R_1(x, y), \dots, R_n(x, y)), F(R_1(y, z), \dots, R_n(y, z))) \leq \\ & F(\min(R_1(x, y), R_1(y, z)), \dots, \min(R_n(x, y), R_n(y, z))) \leq \\ & F(R_1(x, z), \dots, R_n(x, z)) = R(x, z) \end{aligned}$$

which proves the  $\alpha$ -transitivity of a relation  $R_F$  for  $\alpha = F(\alpha_1, \dots, \alpha_n)$ .  $\square$

If we look for functions  $F$  which fulfil both conditions  $F \gg \min$  and  $F \leq \min$  we see that  $F = \min$  which is an aggregation function, fulfils these conditions. Moreover, we have the following property

**Corollary 19** ([18]). For a function  $F : [0, 1]^n \rightarrow [0, 1]$  which has a neutral element  $e = 1$  the following holds true:  $F$  is increasing in each variable,  $F \gg \min$  and  $F \leq \min$  if and only if  $F = \min$ .

It means that the only t-seminorm that fulfils conditions of Corollary 19 is minimum. If it comes to the „converse problem” for  $\alpha$ -transitivity we obtained several counter-examples. In the following example diverse functions were applied to aggregate fuzzy relations, namely greater (smaller) than or equal to minimum (maximum).

**Example 18.** Let card  $X = 3$ . For fuzzy relations described by matrices:

$$R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

we have the following aggregated fuzzy relations

$$\min(R, S) = R \cdot S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\max(R, S) = R + S - R \cdot S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$\frac{R + S}{2} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix},$$

which are  $\alpha$ -transitive for each  $\alpha \in [0, 1]$ , while relations  $R$  and  $S$  do not have this property for any  $\alpha$ . For example for  $\alpha = 1$  and relation  $R$  we have  $\min(r_{12}, r_{21}) = 1 \geq 0$ , but  $0 = r_{11} < \min(r_{12}, r_{21}) = 1$ .

**Remark 5.** Let  $\alpha \in [0, 1]$ . If  $F$  is idempotent, increasing and injective, then if a fuzzy relation  $R_F = F(R_1, \dots, R_n)$  is  $\alpha$ -transitive, then also all relations  $R_1, \dots, R_n$  are  $\alpha$ -transitive. However, these assumptions on  $F$  are contradictory (cf. Remark 4).

## 6. Reciprocity Property and Other Concepts Related to Decision Making Problems

In this section we present notions, concepts and concerns which occur in decision making algorithms.

### 6.1. Reciprocity

Preservation of reciprocity may be useful in aggregation of fuzzy relations. Sometimes this property is required in such situations, so we present adequate assumptions on functions to preserve this property.

**Definition 12** (cf. [4]). Relation  $R \in \mathcal{FR}(X)$  is called reciprocal if for any  $x, y \in X$  it holds  $R(x, y) + R(y, x) = 1$ .

**Definition 13** (cf. [7], p. 31). Let  $F : [0, 1]^n \rightarrow [0, 1]$ . A function  $F^d$  is called a dual function to  $F$ , if for all  $x_1, \dots, x_n \in [0, 1]$

$$F^d(x_1, \dots, x_n) = 1 - F(1 - x_1, \dots, 1 - x_n).$$

$F$  is called a self-dual function, if it holds  $F = F^d$ .

**Theorem 25.** Let  $F : [0, 1]^2 \rightarrow [0, 1]$ .  $F$  is self-dual if and only if  $F$  preserves the reciprocity property of fuzzy relations.

*Proof.* Let  $x, y \in X$ ,  $R_i \in \mathcal{FR}(X)$  for  $i = 1, \dots, n$  be reciprocal fuzzy relations,  $F$  be self-dual, which means that  $F = F^d$ . Thus  $R_i(y, x) = 1 - R_i(x, y)$  and

$$1 - R_F(y, x) = 1 - F(R_1(y, x), \dots, R_n(y, x)) =$$

$$1 - F(1 - R_1(x, y), \dots, 1 - R_n(x, y)) =$$

$$F^d(R_1(x, y), \dots, R_n(x, y)) =$$

$$F(R_1(x, y), \dots, R_n(x, y)) = R_F(x, y). \quad \square$$

The dual functions to fuzzy conjunctions are fuzzy disjunctions and vice versa, and since these two classes are disjoint, it follows that neither a fuzzy conjunction nor a fuzzy disjunction (including t-norms and t-conorms) is a self-dual function. For any binary function  $F$ , if it is without zero divisors (with the zero element 0), then  $F$  cannot be self-dual (see [5]). Any self-dual and commutative binary aggregation function  $F$  satisfies  $F(x, 1 - x) = \frac{1}{2}$  for all  $x \in [0, 1]$ . The concept of self-duality is especially developed for aggregation functions. Interesting properties and characterizations of self-dual aggregation functions one can find in [29]. A weighted arithmetic mean, median and all quasi-linear means for which  $\varphi : [0, 1] \rightarrow [0, 1]$  fulfils  $\varphi(1 - x) = 1 - \varphi(x)$ , are self-dual aggregation functions.

If a relation  $R \in \mathcal{FR}(X)$  is not reciprocal (decision makers were not informed to make such choices) there exist the ways to make it reciprocal. We present such a formula for a finite case, since practically, in decision making problems, we have finite set of alternatives. Let  $R \in \mathcal{FR}(X)$ , where  $X = \{x_1, \dots, x_n\}$ . We may obtain from  $R$  a normalized fuzzy reciprocal relation  $R^* \in \mathcal{FR}(X)$  in the following way

$$R_{ij}^* = \begin{cases} \frac{R_{ij}}{R_{ij} + R_{ji}} & \text{if } R_{ij} + R_{ji} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If we have a reciprocal relation, then we get specific interpretation of properties of this relation (see Section 4, pages 7-8). Reciprocity is in a sense a form of consistency or clearness of choices of decision makers. However, not every function which preserves a given  $\alpha$ -property, preserves also reciprocity. To simplify the considered algorithms, we do not consider the requirement of reciprocity at any stage. We concentrate on  $\alpha$ -properties and their behaviour in aggregation process, which is the main topic of this paper.

## 6.2. Improving Judgements of Decision Makers

It may happen that some of the individual relations will not have the required property, for example  $\alpha$ -transitivity for some  $\alpha$ . In such situation we may assume two options in the algorithms. The first one will be not to run algorithm in such a case. The second one will be to improve a little preferences of decision makers to obtain more 'regular' results, i.e. to obtain all relations with the required property. >From mathematical point of view, if it comes to standard fuzzy relation properties, there are known some results how to improve the relation (for example, to make it transitive, if it is not transitive, cf. [35]). Namely, if relation  $R$  is not reflexive it is enough to consider  $R \vee I$ , where  $I \in \mathcal{FR}(X)$  is the identity relation.  $R \vee I$  is obviously reflexive. If  $R$  is not symmetric we may consider symmetric closure  $R \vee R^{-1}$  or symmetric interior  $R \wedge R^{-1}$ , which are symmetric relations. For asymmetry (antisymmetry) and there is no appropriate closure/interior, so there is no unique method to create the asymmetric (antisymmetric) or connected (total connected) relation from the given one. However, for example the relation  $R \setminus R^{-1}$  is asymmetric. To obtain

the transitive relation from the given  $R$  we may consider its closure as the sum of powers of  $R$ , but often such closure is the full relation ( $R \equiv 1$ ), so it is not useful from practical point of view. However, there are also other methods to obtain a transitive relation from the given non-transitive one, which are not so different from the original relation (cf. [37]). Moreover, the proposed in that paper concept enables to determine relations which are transitive only for a part of the elements under consideration.

If it comes to  $\alpha$ -properties, thanks to the gradualness of these properties we may, in most of the cases, obtain some grade of  $\alpha$  to which a relation has the considered property. We may treat a given  $\alpha$ -property as a measure of this property of the fuzzy relation  $R$ . However, it may happen that we get  $\alpha = 0$ , and in the case of  $\alpha$ -symmetry and  $\alpha$ -transitivity it may be none of  $\alpha \in [0, 1]$  (cf. Remark 3). If we need some improvements of the grade of the given property, we may have two approaches. The first one is to use the existing methods for obtaining standard properties and what is equivalent, in the same way, to obtain  $\alpha$ -properties for any  $\alpha \in [0, 1]$  (cf. Remark 2, Corollary 4). The second one is to increase the grade of  $\alpha$  (not necessarily to the maximum possible value) to which a given relation has the considered  $\alpha$ -property. To change the grade of  $\alpha$  for these properties we may apply Corollaries 5 and 6 in an adequate way (using the properties of infimum and supremum and changing the values of the given relation  $R$ ). Moreover, for reciprocal relations, taking into account properties connected with the diagonal ( $\alpha$ -reflexivity,  $\alpha$ -irreflexivity) there is no need to make any improvements, since by definition, the values on the diagonal are fixed. Furthermore, for reciprocal relations we rather would like to obtain asymmetric than symmetric relations and reciprocal relation is always 0.5-asymmetric and totally 0.5-connected, so practically the following  $\alpha$ -properties may be considered: antisymmetry, connectedness, transitivity.

## 6.3. Methods to Obtain the Final Order of Alternatives

There exist diverse methods to find an alternative as solution from a given  $R \in \mathcal{FR}(X)$ . One of the most widely used is the weighted vote (see [26, 27]). If we have a given preference relation  $R \in \mathcal{FR}(X)$ , where  $X = \{x_1, \dots, x_n\}$ , then the weighted vote strategy means taking as the preferred alternative the solution of

$$\arg \max_{i=1, \dots, n} \sum_{1 \leq j \neq i \leq n} R_{ij}. \quad (20)$$

However, in some situations this method does not allow us to choose an alternative as solution in a unique way (cf. [3]). When this happens, sometimes it is advisable to apply a different method. One of the most widely used methods is the one given by Orlovsky in 1978 and called nondominance method [30]. This method extracts as the solution the least dominated alternative/alternatives of the fuzzy decision making problem starting from a fuzzy preference relation. The maximal nondominated elements of a fuzzy preference relation  $R$  are calculated by means of the follow-

ing operations:

- 1) Compute the fuzzy strict preference relation

$$R_{ij}^s = \begin{cases} R_{ij} - R_{ji} & \text{if } R_{ij} > R_{ji} \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

- 2) Compute the nondominance degree of each alternative  $ND_i = 1 - \bigvee_{j=1, \dots, n} R_{ji}^s$ , so we get a fuzzy set  $ND = [(x_i, ND(x_j)) : x_i \in X]$ .
- 3) Select as alternative:  $arg \max_{i=1, \dots, n} \{ND_i\}$ .

However, with this method we may also not obtain the clear unique result. In such situation there is also a method to obtain an interval-valued fuzzy relation from the given fuzzy relation and then use one of the many possible linear orders for interval-valued fuzzy setting [3], which allows us to obtain the unique alternative from a given set of alternatives  $X$ .

## 7. Comparison of Algorithms

We present here three algorithms to obtain the final solution from a given set of alternatives. We use here theoretical results presented in the paper. Our aim is to compare these approaches for decision making problems. Here, we do not pay attention to the reciprocity requirements (as it was explained before) and ways of obtaining the best alternative. This is why, in the algorithms, we omit this final step of finding the best alternative.

For all presented algorithms, we will have the following inputs:

$X = \{x_1, \dots, x_m\}$ ,  $F$  - aggregation function,  $R_1, \dots, R_n \in \mathcal{FR}(X)$ . The given  $\alpha$ -property will be denoted for short  $\alpha - P$ .

In Algorithm 1 we assume aggregation of fuzzy relations with the common grade of  $\alpha$  for the given property  $\alpha - P$ . Function  $F$  is one of those which preserves such  $\alpha$ -property.

**Algorithm 1** - the steps:

- 1) Check the grade of the property  $\alpha_k - P$  of each  $R_k$  for  $k = 1, \dots, n$
- 2) Fix the common grade of the property  $\alpha - P$  of each  $R_k$  for  $k = 1, \dots, n$
- 3) Determine the relation  $R_F$  with the use of aggregation function  $F$   
Output: the aggregated fuzzy relation  $R_F$  with the property  $\alpha - P$ .

In Algorithm 2 we aggregate fuzzy relations with possible diverse grades of  $\alpha$  for the given property  $\alpha - P$ , i.e.  $\alpha_1 - P$ ,  $\alpha_2 - P$ , ...,  $\alpha_n - P$ . Function  $F$  is one of those which preserves such  $\alpha_1 - P$ ,  $\alpha_2 - P$ , ...,  $\alpha_n - P$  properties.

**Algorithm 2** - the steps:

- 1) Check the grade of the property  $\alpha_k - P$  of each  $R_k$  for  $k = 1, \dots, n$

- 2) Determine the relation  $R_F$  with the use of aggregation function  $F$
- 3) Determine the value  $\alpha = F(\alpha_1, \dots, \alpha_n)$   
Output: the aggregated fuzzy relation  $R_F$  with the property  $\alpha - P$ , where  $\alpha = F(\alpha_1, \dots, \alpha_n)$ .

Note that, if for a given function  $F$  the grade  $\alpha$  in step 3 is different from  $F(\alpha_1, \dots, \alpha_n)$ , then it is enough to put in this step appropriate value of  $\alpha$  (cf. Example 13).

In Algorithm 3 we do not determine the grades of  $\alpha$  for individual fuzzy relations, but we do it for the final result, i.e. the aggregated fuzzy relation. Algorithm 3, with applied appropriate aggregation function  $F$ , guarantees that  $R_1, \dots, R_n$  have the same grade of  $\alpha - P$ , for a given property  $P$ .

**Algorithm 3** - the steps:

- 1) Determine the relation  $R_F$  with the use of aggregation function  $F$
- 2) Check the grade of  $\alpha - P$  for  $R_F$   
Output: the aggregated fuzzy relation  $R_F$  and  $R_1, \dots, R_n$  with the same property  $\alpha - P$ .

In the following subsections we will perform comparison of complexity and usefulness of functions  $F$  preserving diverse properties (more useful practically, weaker assumptions, losing less information etc.).

### 7.1. Comparing Assumptions on Functions Used for Aggregation

We will consider reflexivity property only in the case of general fuzzy relations (not necessarily reciprocal ones). The results for the other properties can be analyzed in a similar way (with similar conclusions). Comparing assumptions on  $F$  for the case of reflexivity we cannot conclude clearly which way is better (Algorithm 1 or Algorithm 2), it depends on the values of fuzzy relations. Let us see some examples.

Let  $\text{card } X = 2$ ,  $R_1, R_2 \in \mathcal{FR}(X)$ ,  $R_F = \frac{R_1 + R_2}{2}$ , where

$$R_1 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.6 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.9 \end{bmatrix},$$

$$R_F = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.75 \end{bmatrix}.$$

$R_1$  is 0.6-reflexive and  $R_2$  is 0.7-reflexive,  $R_F$  is 0.75-reflexive. Considering Algorithm 1, the common value of reflexivity of  $R_1$  and  $R_2$  is 0.6. If we take  $F \geq \min$  (which is for example the arithmetic mean) we have the guarantee that  $F$  preserves 0.6-reflexivity (cf. Theorem 4). However,  $R_F$  may also have the greater level of  $\alpha$ -reflexivity, which is the case for our examples. Considering Algorithm 2, and taking as an aggregating function any increasing  $F$  (cf. Theorem 5) we get for the arithmetic mean  $\alpha = F(0.6, 0.7) = 0.65$ , so  $R_F$  is 0.65-reflexive but it may have higher value of



reflexivity, which is the case in our situation.

Let card  $X = 2$ ,  $R_1, R_2 \in \mathcal{FR}(X)$ ,  $R_F = \frac{R_1 + R_2}{2}$ , where

$$R_1 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.6 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.7 \end{bmatrix},$$

$$R_F = \begin{bmatrix} 0.85 & 0 \\ 0 & 0.65 \end{bmatrix}.$$

$R_1$  is 0.6-reflexive and  $R_2$  is 0.7-reflexive,  $R_F$  is 0.65-reflexive. Considering Algorithm 1, the common value of reflexivity of  $R_1$  and  $R_2$  is 0.6. The arithmetic mean (as explained above) preserves 0.6-reflexivity. However,  $R_F$  may have higher level of  $\alpha$ -reflexivity, which is the case in our example. Considering Algorithm 2, we get for the arithmetic mean  $\alpha = F(0.6, 0.7) = 0.65$ , so  $R_F$  is 0.65-reflexive and this coincides with the real value of reflexivity in the considered example.

If it comes to Algorithm 3, it is enough to check the grade of  $\alpha$ -reflexivity of the fuzzy relation  $R_F$  and if aggregating function fulfils the property  $F \leq \min$ , then we know that all aggregated relations  $R_1, \dots, R_n$  are of the same grade of  $\alpha$ -reflexivity (cf. Theorem 6). There is also the risk of loosing information about the real grade of  $\alpha$  of particular fuzzy relations involved in the process of aggregation. Let us see the example, where  $R_1, R_2 \in \mathcal{FR}(X)$ ,  $F = \min$  and

$$R_1 = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$R_F = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

$R_F$  and  $R_1, R_2$  are 0.2-reflexive, but  $R_1$  is in fact 0.6-reflexive.

To sum up, using these methods we should remember that the grade of  $\alpha$ -property of the aggregated fuzzy relation  $R_F$  (Algorithms 1 and 2) and input fuzzy relations  $R_1, \dots, R_n$  (Algorithm 3) 'is the minimal of the maximum possible' to be obtained (all depends on the form of fuzzy relations).

It is also worth mentioning that assumptions on functions  $F$  to preserve  $\alpha$ -transitivity are rather strong. However, if in definition of  $\alpha$ -transitivity we replace  $\min$  with arbitrary binary operation  $*$ :  $[0, 1]^2 \rightarrow [0, 1]$ , then we may weaken in a significant way the assumptions on function  $F$  to preserve such  $\alpha$ -\*transitivity. Thus it is enough if  $F \gg *$  and  $F \leq \min$ , and we have many examples of such functions  $F$  (cf. [17]).

For aggregation of fuzzy relations with diverse grades, some assumptions seem to be strong. In Theorem 17 for preservation of asymmetry we have strong assumption  $F \gg \min$ , but for concrete functions  $F$ , like for example the arithmetic mean in Example 13, we may compute the value of  $\alpha$  for the property  $\alpha$ -P (without following assumptions of Theorem 17). Note that in this example  $\alpha \neq F(\alpha_1, \dots, \alpha_n)$ .

If it comes to the converse problem, for  $\alpha$ -symmetry and  $\alpha$ -transitivity it is not clear if such functions do exist (cf. Remarks 4 and 5).

## 7.2. Comparison of Complexity of the Algorithms

We will present the time complexity of the separate steps and operations in the presented algorithms and then we will give the complexity of each algorithm for each property.

Fuzzy relations  $R_1, \dots, R_n$  are defined in a set  $X$  consisting of  $m$  elements, so complexity will depend on the variable  $m$  (the size of a matrix representing  $R$ ). We get the following time complexities:

- determining the grade  $\alpha$  of reflexivity (irreflexivity) is  $O(m)$ , since this is determining the minimal (maximal) value of a list of  $m$  non-ordered elements (cf. Corollary 5),
- determining the grade  $\alpha$  of connectedness (total connectedness, asymmetry, antisymmetry) is  $O(m^2)$  (cf. Corollary 5),
- determining the grade  $\alpha$  of symmetry is  $O(m^2)$  (cf. Corollary 6),
- determining the grade  $\alpha$  of transitivity is  $O(m^3)$  (cf. Corollary 6).

Taking into account that the remaining operations in Algorithms 1, 2, 3 should be performed at maximum  $n$  times ( $n$  is the number of fuzzy relations) we get the following time complexities.

**Corollary 20.** *Reflexivity and irreflexivity: Algorithms 1, 2 and 3 take  $O(m)$  computational time complexity. Connectedness, total connectedness, symmetry, asymmetry and antisymmetry: Algorithms 1, 2 and 3 take  $O(m^2)$  computational time complexity. Transitivity: Algorithms 1, 2 and 3 take  $O(m^3)$  computational time complexity.*

## 8. Conclusion

In this paper preservation of basic classes of  $\alpha$ -properties of fuzzy relations in the context of aggregation process were discussed. Mutual dependencies related to these properties, between relations  $R_1, \dots, R_n$  on a set  $X$  and the aggregated fuzzy relation  $R_F = F(R_1, \dots, R_n)$  were examined. Sufficient conditions for functions  $F: [0, 1]^n \rightarrow [0, 1]$  to fulfill the given property were provided (regarding three possible cases of approach to aggregation procedure). Moreover, diverse 'regularities' and interpretation of  $\alpha$ -properties were discussed, also in the context of reciprocal relations and decision making problems. Finally, comparison of obtained results, including suitable decision making algorithms were provided (there were analyzed the time complexities of the presented algorithms and assumptions on fusion functions useful to obtain the required results). All algorithms were implemented and tested in Java programming language.

## ACKNOWLEDGEMENTS

This work was partially supported by the Centre for Innovation and Transfer of Natural Sciences and Engineering Knowledge in Rzeszów, through Project Number RPPK.01.03.00-18-001/10.

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