A Novel Approach to the Solution of Matrix Games with Payoffs Expressed by Trapezoidal Intuitionistic Fuzzy Numbers

Submitted: 22th April 2015; accepted 26th May 2015

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DOI: 10.14313/JAMRIS_3-2015/22

Abstract:
We propose a novel approach to the solution of fuzzy matrix games with payoffs given as trapezoidal intuitionistic fuzzy numbers. We extend Li's [36, Chapter 9] work based on a cut-set based method for finding an optimal solution to overcome the fact that the assumptions and properties assumed therein do not guarantee in general, first, the very existence of an optimal solution, and second, its attainment via a mathematical programming formulation proposed. We first briefly mention those problems in Li's [36] approach, and then propose a new, corrected and general method, called the Mehar mehod, based on a modified mathematical programming formulation of a matrix game with payoffs represented by trapezoid intuitionistic fuzzy numbers. For illustration, we solve Li's [36] example, and compare his and our results.

Keywords: *trapezoidal intuitionistic fuzzy numbers, mathematical programming problem, two person zero sum game, α cut, β cut*

1. Introduction

Game theory provide many effective and efficient tools and techniques to mathematically formulate and solve many multiperson (collective, multiagent …) with strategic interactions among multiple rational decision makers (cf. von Neumann and Morgenstern [56]); for various kinds of mathematical formulations and solution concepts, cf. [61].

Usually, the classical, i.e. nonfuzzy or crisp in our context, game theory assumes the payoffs of players to be real numbers. In reality, we often need to represent the players' payoffs by their subjective judgments (or opinions) so that natural language descriptions using terms such as "very large", "larger", "medium", "small", "smaller than" may be more adequate. Obviously, they are inherently imprecise and fuzzy sets theory [72] can provide effective and efficient tools and techniques.

In fuzzy sets, that imprecision is handled by assigning a degree, from [0,1], called the membership degree, to which an object belongs to a set; the degree to which it does not belong to the same set, the non-membership degree is one minus the membership degree.

However, in many human centered/focused real decisions making problems such a simple representation of imprecision is not sufficient to adequately represent judgments by (human) decision makers. Basically, the human beings often tend to provide arguments "pro", i.e. somehow taking into account "good" aspects, and "con", i.e. taking into account "bad" aspects. The arguments in favor of "good" imply the membership degree that the product belongs to a "set of good products", while the arguments in favor of 'badness' imply the non-membership degree of the product in the "set of good products". Moreover, a human being may have his/her own reservations in classifying the object into one of these two categories, "good" and "bad", so that the two degrees do not necessarily add up to one. Atanassov [3] proposed an interesting generalization of fuzzy sets called intuitionistic fuzzy sets to capture this aspect of human judgments and behavior. In the intuitionistic fuzzy sets, we have two membership functions, one that describes the degree of belongingness of elements and the other that describes the degree of non-belongingness, and the sum of those degrees is less than or equal to one.

Over the last decades both the fuzzy sets and the intuitionistic fuzzy sets theories have enjoyed much popularity, and – which is important for our purposes – they have been applied in various game theoretic contexts, for instance, cf. [1, 2, 4–19, 21–24, 27–55, 57–60, 62–71].

2. A Brief Review of Approaches to Non-cooperative Games with Payoffs Represented by Interval/Fuzzy Numbers/ Intuitionistic Fuzzy Numbers

Games may be classified into two major categories: cooperative and non-cooperative. Though the cooperation of players may be assumed in many games, the existence of non-cooperation is probably more attractive because it is often more realistic, especially upon the competition between players. In the non-cooperative games, an important, from conceptual and application points of view, class of games are matrix games (or two-person zero-sum games).

In this section, we will provide a brief review of some recent works, which are relevant for this paper, dealing with non-cooperative games in which payoffs are represented by interval/fuzzy numbers/ intuitionistic fuzzy numbers, with a mathematical programming problem formulation; this will also be assumed here.

Li and Cheng [37] transformed the fuzzy linear programming problem representing such fuzzy matrix games in which payoffs are represented by triangular fuzzy numbers, into a crisp linear programming problem and used the crisp optimal solution obtained to get the crisp optimal solution and a fuzzy optimal value of the fuzzy constrained matrix game.

Bector et al. [7] transformed the fuzzy linear programming problem of such fuzzy matrix games in which payoffs are represented by triangular fuzzy numbers, into a crisp linear programming problem and used the crisp optimal solution obtained toget the fuzzy optimal solution of the fuzzy matrix game.

Liu and Kao [46] transformed the fuzzy linear programming problem of such fuzzy matrix games with triangular fuzzy number payoffs into a pair of two-level mathematical programming problems and obtained, by solving them, the lower and upper bound of the optimal fuzzy value of the fuzzy matrix game.

Li [30] transformed the fuzzy linear programming problem of such fuzzy matrix games in with triangular fuzzy number payoffs into two crisp linear programming problems and used the crisp optimal solution, obtained by the lexicographic method, to get the fuzzy optimal solution of the fuzzy matrix games.

Liu and Kao [47] transformed the interval linear programming problem of such matrix games in which payoffs are represented by intervals, into a pair of two level mathematical programming problems and obtained, by solving them, the lower and upper bound of the optimal interval value of the interval matrix game.

Nan et al. [53] transformed the intuitionistic fuzzy linear programming problem of such intuitionistic fuzzy matrix games with triangular intuitionistic fuzzy number payoffs into two crisp linear programming problems and used the crisp optimal solution, obtained by the lexicographic method, to get the intuitionistic fuzzy optimal solution of the intuitionistic fuzzy matrix games.

Li [32] transformed the interval linear programming problem of such matrix games with payoffs represented by intervals, into two crisp linear programming problems and obtained, by solving them, the lower and upper bound of the optimal interval value of the interval matrix game.

Li [34] transformed the fuzzy linear programming problem of such fuzzy matrix games with triangular fuzzy number payoffs, into three crisp linear programming problems and used the crisp optimal solutions obtained to get the fuzzy optimal solution of the fuzzy matrix games.

Li and Hong [38] transformed the fuzzy linear programming problem of such fuzzy constrained matrix games with triangular fuzzy number payoffs into three crisp linear programming problems and used the crisp optimal solutions obtained to get the fuzzy optimal solution and the fuzzy optimal value of the fuzzy constrained matrix game.

Li and Hong [39] transformed the fuzzy linear programming problem of such fuzzy constrained matrix games with trapezoidal fuzzy number payoffs into four crisp linear programming problems and used the crisp optimal solutions obtained to get the fuzzy optimal solution and the fuzzy optimal value of the fuzzy constrained matrix game.

Li et al. [41] transformed the fuzzy linear programming problem of such fuzzy matrix games with triangular intutionistic fuzzy number payoffs into a crisp linear programming problem and used the crisp optimal solutions obtained to get the crisp optimal solution of the fuzzy matrix game.

Li [35] transformed the fuzzy linear programming problem of such fuzzy matrix games with trapezoidal fuzzy number payoffs into four crisp linear programming problems and used the crisp optimal solutions obtained to get the fuzzy optimal solution of the fuzzy matrix games.

Li and Yang [43] transformed the intuitionistic fuzzy bilinear programming problem of such intuitionistic fuzzy bimatrix games with trapezoidal intutionistic fuzzy number payoffs into a crisp bilinear programming problem and used the crisp optimal solution obtained to get the crisp optimal solution and the intuitionistic fuzzy optimal value of the intutitionistic fuzzy bimatrix games.

Nan et al. [54] transformed the intuitionistic fuzzy linear programming problem of such intuitionistic fuzzy matrix games with triangular intuitionistic fuzzy number payoffs into two crisp linear programming problems and used the crisp optimal solutions obtained to get the crisp optimal solution and the intuitionistic fuzzy optimal value of the intuitionistic fuzzy matrix games.

Li [36, Chapter 9, Section 9.3] proposed a cut set based method of such intuitionistic fuzzy matrix games with trapezoidal intuitionistic fuzzy number payoffs by transforming the intuitionistic fuzzy linear programming problem into a crisp linear programming problem and used the crisp optimal solution obtained to get the intuitionistic fuzzy optimal solution of the intuitionistic fuzzy matrix games.

This paper is basically along the lines of that Li's [36] work, and a novel method will be proposed to alleviate some shortcomings of his method and provide generality. More specifically, to resolve these shortcomings, a correct mathematical formulation of such matrix games with trapezoidal intuitionistic fuzzy number payoffs is developed and a new method (to be called the Mehar mehod) to find the exact optimal solution is proposed.

3. The Existing Method

Basically, Li [36] proposed the following method to find the optimal solution of matrix games in which payoffs are represented by trapezoidal intuitionistic fuzzy numbers:

Step 1: Formulate the problem considered as the following mathematical programming problem:

Problem P1

Maximize $(\tilde{v})$

Subject to:

$$\sum_{i=1}^{m} \tilde{a}_{ij} x_i \tilde{\geq} \tilde{v},\ j = 1,2,\ldots,n;$$

$$\sum_{i=1}^{m} x_i = 1;$$

$$x_i \geq 0,\ i = 1,2,\ldots,m.$$

Step 2: Since the $\tilde{a}_{ij}$'s are known, them by assuming $\tilde{a}_{ij} = \left\langle \left(\underline{a}_{ij}, a_{ij}^L, a_{ij}^U, \overline{a}_{ij}\right); w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle$, problem P1 can be transformed into the following problem P2.

Problem P2

Maximize $(\tilde{v})$
Subject to

$$\sum_{i=1}^{m} \left(\left\langle \left(\underline{a}_{ij}, a_{ij}^L, a_{ij}^U, \overline{a}_{ij}\right); w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle \right) x_i \tilde{\geq} \tilde{v},\ j = 1,2,\ldots,n;$$

$$\sum_{i=1}^{m} x_i = 1;$$

$$x_i \geq 0, \quad i = 1,2,\ldots,m.$$

Step 3: Using the property $\tilde{a} \tilde{\geq} \tilde{b} \Rightarrow (\tilde{a})_\alpha \geq (\tilde{b})_\alpha$ and $(\tilde{a})^\beta \geq (\tilde{b})^\beta$, problem P2 can be transformed into the following biobjective mathematical programming problem:

Problem P3

Maximize $(\tilde{v}_\alpha)$, Maximize $(\tilde{v}^\beta)$
Subject to

$$\left(\sum_{i=1}^{m} \left(\left\langle \left(\underline{a}_{ij}, a_{ij}^L, a_{ij}^U, \overline{a}_{ij}\right); w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle \right) x_i \right)_\alpha \geq \tilde{v}_\alpha,\ j = 1,2,\ldots,n;$$

$$\left(\sum_{i=1}^{m} \left(\left\langle \left(\underline{a}_{ij}, a_{ij}^L, a_{ij}^U, \overline{a}_{ij}\right); w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle \right) x_i \right)^\beta \geq \tilde{v}^\beta,\ j = 1,2,\ldots,n;$$

$$\sum_{i=1}^{m} x_i = 1;$$

$$x_i \geq 0, \quad i = 1,2,\ldots,m.$$

where, $\alpha \in \left[0, \min_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} \left\{ w_{\tilde{a}_{ij}} \right\} \right], \beta \in \left[\max_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} \left\{ u_{\tilde{a}_{ij}} \right\}, 1 \right].$

Step 4: Since, obviously: $\left(\sum_{i=1}^{n} \tilde{a}_i x_i \right)_\alpha = \sum_{i=1}^{n} (\tilde{a}_i)_\alpha x_i$ and $\left(\sum_{i=1}^{n} \tilde{a}_i x_i \right)^\beta = \sum_{i=1}^{n} (\tilde{a}_i)^\beta x_i$, problem P3 can be transformed into:

Problem P4

Maximize $(\tilde{v}_\alpha)$, Maximize $(\tilde{v}^\beta)$
Subject to:

$$\sum_{i=1}^{m} \left(\left\langle \left(\underline{a}_{ij}, a_{ij}^L, a_{ij}^U, \overline{a}_{ij}\right); w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle \right)_\alpha x_i \geq \tilde{v}_\alpha,\ j = 1,2,\ldots,n;$$

$$\sum_{i=1}^{m} \left(\left\langle \left(\underline{a}_{ij}, a_{ij}^L, a_{ij}^U, \overline{a}_{ij}\right); w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle \right)^\beta x_i \geq \tilde{v}^\beta,\ j = 1,2,\ldots,n;$$

$$\sum_{i=1}^{m} x_i = 1;$$

$$x_i \geq 0, \quad i = 1,2,\ldots,m.$$

where, $\alpha \in \left[0, \min_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} \left\{ w_{\tilde{a}_{ij}} \right\} \right], \beta \in \left[\max_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} \left\{ u_{\tilde{a}_{ij}} \right\}, 1 \right].$

Step 5: Using the values $\left(\left\langle\left(\underline{a}_i, a_i^L, a_i^U, \overline{a}_i\right); w_{\tilde{a}_i}, u_{\tilde{a}_i}\right\rangle\right)_\alpha = \left[\frac{\left(w_{\tilde{a}_i}-\alpha\right)\underline{a}_i+\alpha a_i^L}{w_{\tilde{a}_i}}, \frac{\left(w_{\tilde{a}_i}-\alpha\right)\overline{a}_i+\alpha a_i^U}{w_{\tilde{a}_i}}\right]$,

$\left(\left\langle\left(\underline{a}_i, a_i^L, a_i^U, \overline{a}_i\right); w_{\tilde{a}_i}, u_{\tilde{a}_i}\right\rangle\right)^\beta = \left[\frac{(1-\beta)a_i^L+\left(\beta-u_{\tilde{a}_i}\right)\underline{a}_i}{1-u_{\tilde{a}_i}}, \frac{(1-\beta)a_i^U+\left(\beta-u_{\tilde{a}_i}\right)\overline{a}_i}{1-u_{\tilde{a}_i}}\right]$ and assuming $\tilde{v}_\alpha = \left[v_\alpha^L, v_\alpha^R\right]$ and $\tilde{v}^\beta = \left[v_L^\beta, v_R^\beta\right]$, problem P4 can be transformed into:

Problem P5

Maximize $\left(\left[v_\alpha^L, v_\alpha^R\right]\right)$, Maximize $\left(\left[v_L^\beta, v_R^\beta\right]\right)$
Subject to

$$\sum_{i=1}^{m}\left[\frac{\left(w_{\tilde{a}_{ij}}-\alpha\right)\underline{a}_{ij}+\alpha a_{ij}^L}{w_{\tilde{a}_{ij}}}, \frac{\left(w_{\tilde{a}_{ij}}-\alpha\right)\overline{a}_{ij}+\alpha a_{ij}^U}{w_{\tilde{a}_{ij}}}\right]x_i \geq \left[v_\alpha^L, v_\alpha^R\right], j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left[\frac{(1-\beta)a_{ij}^L+\left(\beta-u_{\tilde{a}_{ij}}\right)\underline{a}_{ij}}{1-u_{\tilde{a}_{ij}}}, \frac{(1-\beta)a_{ij}^U+\left(\beta-u_{\tilde{a}_{ij}}\right)\overline{a}_{ij}}{1-u_{\tilde{a}_{ij}}}\right]x_i \geq \left[v_L^\beta, v_R^\beta\right], j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m} x_i = 1;$$

$$x_i \geq 0, \quad i=1,2,\ldots,m.$$

where, $\alpha \in \left[0, \min_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{w_{\tilde{a}_{ij}}\right\}\right], \beta \in \left[\max_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{u_{\tilde{a}_{ij}}\right\}, 1\right]$.

Step 6: Using the property $[a,b]x=[ax,bx]; \quad x\geq 0$, problem P5 can be transformed into:

Problem P6

Maximize $\left(\left[v_\alpha^L, v_\alpha^R\right]\right)$, Maximize $\left(\left[v_L^\beta, v_R^\beta\right]\right)$
Subject to:

$$\sum_{i=1}^{m}\left[\left(\frac{\left(w_{\tilde{a}_{ij}}-\alpha\right)\underline{a}_{ij}+\alpha a_{ij}^L}{w_{\tilde{a}_{ij}}}\right)x_i, \left(\frac{\left(w_{\tilde{a}_{ij}}-\alpha\right)\overline{a}_{ij}+\alpha a_{ij}^U}{w_{\tilde{a}_{ij}}}\right)x_i\right] \geq \left[v_\alpha^L, v_\alpha^R\right], j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left[\left(\frac{(1-\beta)a_{ij}^L+\left(\beta-u_{\tilde{a}_{ij}}\right)\underline{a}_{ij}}{1-u_{\tilde{a}_{ij}}}\right)x_i, \left(\frac{(1-\beta)a_{ij}^U+\left(\beta-u_{\tilde{a}_{ij}}\right)\overline{a}_{ij}}{1-u_{\tilde{a}_{ij}}}\right)x_i\right] \geq \left[v_L^\beta, v_R^\beta\right], \qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m} x_i = 1;$$

$$x_i \geq 0, \quad i=1,2,\ldots,m.$$

where, $\alpha \in \left[0, \min_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{w_{\tilde{a}_{ij}}\right\}\right], \beta \in \left[\max_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{u_{\tilde{a}_{ij}}\right\}, 1\right]$.

Step 7: Using the property $\sum_{i=1}^{n}[a_i, b_i] = \left[\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i\right]$, problem P6 can be transformed into:

Problem P7

Maximize $\left(\left[v_\alpha^L, v_\alpha^R\right]\right)$, Maximize $\left(\left[v_L^\beta, v_R^\beta\right]\right)$
Subject to

$$\left[\sum_{i=1}^{m}\left(\frac{\left(w_{\tilde{a}ij}-\alpha\right)\underline{a}_{ij}+\alpha a_{ij}^L}{w_{\tilde{a}ij}}\right)x_i,\sum_{i=1}^{m}\left(\frac{\left(w_{\tilde{a}ij}-\alpha\right)\overline{a}_{ij}+\alpha a_{ij}^U}{w_{\tilde{a}ij}}\right)x_i\right]\geq\left[v_\alpha^L,v_\alpha^R\right],\qquad j=1,2,\ldots,n;$$

$$\left[\sum_{i=1}^{m}\left(\frac{(1-\beta)a_{ij}^L+\left(\beta-u_{\tilde{a}ij}\right)\underline{a}_{ij}}{1-u_{\tilde{a}ij}}\right)x_i,\sum_{i=1}^{m}\left(\frac{(1-\beta)a_{ij}^U+\left(\beta-u_{\tilde{a}ij}\right)\overline{a}_{ij}}{1-u_{\tilde{a}ij}}\right)x_i\right]\geq\left[v_L^\beta,v_R^\beta\right], j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}x_i=1;$$

$$x_i\geq 0,\quad i=1,2,\ldots,m.$$

where, $\alpha\in\left[0,\min_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{w_{\tilde{a}ij}\right\}\right],\beta\in\left[\max_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{u_{\tilde{a}ij}\right\},1\right]$.

Step 8: Using the property $[a,b]\geq[c,d]\Rightarrow a\geq c, b\geq d$, problem P7 can be transformed into:

Problem P8

Maximize $\left(\left[v_\alpha^L, v_\alpha^R\right]\right)$, Maximize $\left(\left[v_L^\beta, v_R^\beta\right]\right)$
Subject to:

$$\sum_{i=1}^{m}\left(\frac{\left(w_{\tilde{a}ij}-\alpha\right)\underline{a}_{ij}+\alpha a_{ij}^L}{w_{\tilde{a}ij}}\right)x_i\geq v_\alpha^L,\qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\frac{\left(w_{\tilde{a}ij}-\alpha\right)\overline{a}_{ij}+\alpha a_{ij}^U}{w_{\tilde{a}ij}}\right)x_i\geq v_\alpha^R,\qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\frac{(1-\beta)a_{ij}^L+\left(\beta-u_{\tilde{a}ij}\right)\underline{a}_{ij}}{1-u_{\tilde{a}ij}}\right)x_i\geq v_L^\beta,\qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\frac{(1-\beta)a_{ij}^U+\left(\beta-u_{\tilde{a}ij}\right)\overline{a}_{ij}}{1-u_{\tilde{a}ij}}\right)x_i\geq v_R^\beta,\qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}x_i=1;$$

$$x_i\geq 0,\quad i=1,2,\ldots,m.$$

where, $\alpha\in\left[0,\min_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{w_{\tilde{a}ij}\right\}\right],\beta\in\left[\max_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{u_{\tilde{a}ij}\right\},1\right]$.

Step 9: The two interval-valued objective functions in problem P8 may be regarded to be of equal importance, i.e., with weights of 0.5. Therefore, using the linear weighted averaging method of the multi-objective decision making as proposed in [20, 25, 28], problem P8 can be aggregated into the following interval-valued mathematical programming problem:

Problem P9

Maximize $\left(\left[\frac{v_\alpha^L+v_L^\beta}{2},\frac{v_\alpha^R+v_R^\beta}{2}\right]\right)$

Subject to:

$$\sum_{i=1}^{m}\left(\frac{\left(w_{\tilde{a}ij}-\alpha\right)\underline{a}_{ij}+\alpha a_{ij}^{L}}{w_{\tilde{a}ij}}\right)x_i \geq v_{\alpha}^{L}, \qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\frac{\left(w_{\tilde{a}ij}-\alpha\right)\overline{a}_{ij}+\alpha a_{ij}^{U}}{w_{\tilde{a}ij}}\right)x_i \geq v_{\alpha}^{R}, \qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\frac{(1-\beta)a_{ij}^{L}+\left(\beta-u_{\tilde{a}ij}\right)\underline{a}_{ij}}{1-u_{\tilde{a}ij}}\right)x_i \geq v_{L}^{\beta}, \qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\frac{(1-\beta)a_{ij}^{U}+\left(\beta-u_{\tilde{a}ij}\right)\overline{a}_{ij}}{1-u_{\tilde{a}ij}}\right)x_i \geq v_{R}^{\beta}, \qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}x_i=1;$$

$$x_i \geq 0, \quad i=1,2,\ldots,m.$$

where, $\alpha\in\left[0,\min_{\substack{1\le i\le m\\1\le j\le n}}\left\{w_{\tilde{a}ij}\right\}\right],\beta\in\left[\max_{\substack{1\le i\le m\\1\le j\le n}}\left\{u_{\tilde{a}ij}\right\},1\right]$.

Step 10: According to Ishibushi and Tanaka [26], the interval-valued objective function $[a,b]$ is equivalent to the biobjective performance (objective) function $\left[a,\frac{a+b}{2}\right]$. So, problem P9 can be transformed into:

Problem P10

Maximize $\left(\left[\frac{v_{\alpha}^{L}+v_{L}^{\beta}}{2},\frac{v_{\alpha}^{L}+v_{L}^{\beta}+v_{\alpha}^{R}+v_{R}^{\beta}}{4}\right]\right)$

Subject to:

$$\sum_{i=1}^{m}\left(\frac{\left(w_{\tilde{a}ij}-\alpha\right)\underline{a}_{ij}+\alpha a_{ij}^{L}}{w_{\tilde{a}ij}}\right)x_i \geq v_{\alpha}^{L}, \qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\frac{\left(w_{\tilde{a}ij}-\alpha\right)\overline{a}_{ij}+\alpha a_{ij}^{U}}{w_{\tilde{a}ij}}\right)x_i \geq v_{\alpha}^{R}, \qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\frac{(1-\beta)a_{ij}^{L}+\left(\beta-u_{\tilde{a}ij}\right)\underline{a}_{ij}}{1-u_{\tilde{a}ij}}\right)x_i \geq v_{L}^{\beta}, \qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\frac{(1-\beta)a_{ij}^{U}+\left(\beta-u_{\tilde{a}ij}\right)\overline{a}_{ij}}{1-u_{\tilde{a}ij}}\right)x_i \geq v_{R}^{\beta}, \qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}x_i=1;$$

$$x_i \geq 0, \quad i=1,2,\ldots,m.$$

where, $\alpha\in\left[0,\min_{\substack{1\le i\le m\\1\le j\le n}}\left\{w_{\tilde{a}ij}\right\}\right],\beta\in\left[\max_{\substack{1\le i\le m\\1\le j\le n}}\left\{u_{\tilde{a}ij}\right\},1\right]$.

Step 11: Using the usual linear weighted average, cf. [20, 25, 28], problem P10 can be transformed into:

Problem P11

Maximize $\left(\frac{1}{2}\left(\frac{v_\alpha^L+v_L^\beta}{2}\right)+\frac{1}{2}\left(\frac{v_\alpha^L+v_L^\beta+v_\alpha^R+v_R^\beta}{4}\right)\right)$

Subject to

$$\sum_{i=1}^{m}\left(\frac{\left(w_{\tilde{a}ij}-\alpha\right)\underline{a}_{ij}+\alpha a_{ij}^L}{w_{\tilde{a}ij}}\right)x_i \geq v_\alpha^L,\qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\frac{\left(w_{\tilde{a}ij}-\alpha\right)\overline{a}_{ij}+\alpha a_{ij}^U}{w_{\tilde{a}ij}}\right)x_i \geq v_\alpha^R,\qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\frac{(1-\beta)a_{ij}^L+\left(\beta-u_{\tilde{a}ij}\right)\underline{a}_{ij}}{1-u_{\tilde{a}ij}}\right)x_i \geq v_L^\beta,\qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\frac{(1-\beta)a_{ij}^U+\left(\beta-u_{\tilde{a}ij}\right)\overline{a}_{ij}}{1-u_{\tilde{a}ij}}\right)x_i \geq v_R^\beta,\qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}x_i=1;$$

$$x_i \geq 0,\quad i=1,2,\ldots,m.$$

where, $\alpha\in\left[0,\min_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{w_{\tilde{a}ij}\right\}\right], \beta\in\left[\max_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{u_{\tilde{a}ij}\right\},1\right]$.

Step 12: Find the optimal solution $\left\{v_\alpha^L,v_L^\beta,v_\alpha^R,v_R^\beta,x_i;i=1,2,\ldots,m\right\}$ of problem P11 for some values of the parameters $\alpha\in\left[0,\min_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{w_{\tilde{a}ij}\right\}\right]$ and $\beta\in\left[\max_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{u_{\tilde{a}ij}\right\},1\right]$, where $0\leq\alpha+\beta\leq 1$.

Step 13: Using the optimal solution obtained in Step 12, the $\alpha-$ cut and $\beta-$cut of the intuitionistic fuzzy optimal value of problem P1 corresponding to the chosen values of α and β are $\left[v_\alpha^L,v_\alpha^R\right]$ and $\left[v_L^\beta,v_R^\beta\right]$, respectively.

4. Remark on Some Flaws in the Existing Method

Basically, there are the following flaws in the existing Li's [36, Chapter 9, Section 9.3.1, pp. 361] method, which is mainly related to the fact that the properties therein used do not hold in general:

1. In Step 4 of Li's method, described in Section 3, for transforming problem P3 into problem P4, the author has used the mathematical properties

$$\left(\sum_{i=1}^{n}\tilde{a}_i\right)_\alpha=\sum_{i=1}^{n}\left(\tilde{a}_i\right)_\alpha \text{ and } \left(\sum_{i=1}^{n}\tilde{a}_i\right)^\beta=\sum_{i=1}^{n}\left(\tilde{a}_i\right)^\beta.$$

However, it is obvious from (1) and (2) that, in general, $\left(\sum_{i=1}^{n}\tilde{a}_i\right)_\alpha\neq\sum_{i=1}^{n}\left(\tilde{a}_i\right)_\alpha$ since: $\left(\sum_{i=1}^{n}\tilde{a}_i\right)_\alpha$

$$=\left(\sum_{i=1}^{n}\left\langle\left(\underline{a}_i,a_i^L,a_i^U,\overline{a}_i\right);w_{\tilde{a}i},u_{\tilde{a}i}\right\rangle\right)_\alpha$$

$$=\left(\left\langle\left(\sum_{i=1}^{n}\underline{a}_i,\sum_{i=1}^{n}a_i^L,\sum_{i=1}^{n}a_i^U,\sum_{i=1}^{n}\overline{a}_i\right);\min_{1\leq i\leq n}\left\{w_{\tilde{a}i}\right\},\max_{1\leq i\leq n}\left\{u_{\tilde{a}i}\right\}\right\rangle\right)_\alpha$$

$$=\left[\frac{\left(\min\limits_{1\le i\le n}\left\{w_{\tilde{a}_i}\right\}-\alpha\right)\sum\limits_{i=1}^{n}\underline{a}_i+\alpha\sum\limits_{i=1}^{n}a_i^L}{\min\limits_{1\le i\le n}\left\{w_{\tilde{a}_i}\right\}},\frac{\left(\min\limits_{1\le i\le n}\left\{w_{\tilde{a}_i}\right\}-\alpha\right)\sum\limits_{i=1}^{n}\overline{a}_i+\alpha\sum\limits_{i=1}^{n}a_i^U}{\min\limits_{1\le i\le n}\left\{w_{\tilde{a}_i}\right\}}\right] \quad (1)$$

and

$$\sum_{i=1}^{n}(\tilde{a}_i)_\alpha$$
$$=\sum_{i=1}^{n}\left(\left\langle\left(\underline{a}_i,a_i^L,a_i^U,\overline{a}_i\right);w_{a_i},u_{a_i}\right\rangle\right)_\alpha$$
$$=\sum_{i=1}^{n}\left[\frac{\left(w_{\tilde{a}_i}-\alpha\right)\underline{a}_i+\alpha a_i^L}{w_{\tilde{a}_i}},\frac{\left(w_{\tilde{a}_i}-\alpha\right)\overline{a}_i+\alpha a_i^U}{w_{\tilde{a}_i}}\right]$$
$$=\left[\sum_{i=1}^{n}\left(\frac{\left(w_{\tilde{a}_i}-\alpha\right)\underline{a}_i+\alpha a_i^L}{w_{\tilde{a}_i}}\right),\sum_{i=1}^{n}\left(\frac{\left(w_{\tilde{a}_i}-\alpha\right)\overline{a}_i+\alpha a_i^U}{w_{\tilde{a}_i}}\right)\right] \quad (2)$$

2. In the interval $[a,b]$, the inequality $a\le b$ should always be satisfied. However, in Step 12 problem P11 is solved without the restrictions $v_\alpha^R\ge v_\alpha^L$ and $v_R^\beta\ge v_L^\beta$. So, for the obtained values of v_α^R,v_α^L, v_R^β,v_L^β the inequalities $v_\alpha^R\ge v_\alpha^L$ and $v_R^\beta\ge v_L^\beta$ may or may not be satisfied, in general.

Li [36] solves problem P12 (and problem P13) to illustrate his proposed method and obtains the optimal solution as shown in Table 1.

Problem P12

Maximize $(\tilde{v})$
Subject to

$$\langle(175,180,190);0.6,0.2\rangle x_1+\langle(80,90,100);0.9,0.1\rangle x_2\,\tilde{\ge}\,\tilde{v};$$
$$\langle(150,156,158);0.6,0.1\rangle x_1+\langle(175,180,190);0.6,0.2\rangle x_2\,\tilde{\ge}\,\tilde{v};$$
$$x_1+x_2=1;$$
$$x_1,x_2\ge 0.$$

Table 1. Maximin strategies and the cut sets for specific values of the ordered pair $\langle\alpha,\beta\rangle$ obtained by Li's method

$\langle\alpha,\beta\rangle$	$x^*(\alpha,\beta)$	$\tilde{v}^*_{\langle\alpha,\beta\rangle}$
$\langle 0,1\rangle$	$(0.792,0.208)$	$[155.2,164.7]$
$\langle 0.1,0.8\rangle$	$(0.792,0.208)$	$[156.5,163.8]$
$\langle 0.2,0.7\rangle$	$(0.793,0.207)$	$[157.2,163.3]$
$\langle 0.3,0.6\rangle$	$(0.794,0.206)$	$[158.1,162.8]$
$\langle 0.4,0.5\rangle$	$(0.817,0.183)$	$[158.4,161.5]$
$\langle 0.5,0.3\rangle$	$(0.795,0.205)$	$[160.0,161.5]$
$\langle 0.6,0.2\rangle$	$(0.795,0.205)$	160.9

It is obvious from Table 1 that the values of x_1 and x_2 vary with the change of α and β which indicates that x_1 and x_2 should rather be meant, in our context, as intuitionistic fuzzy numbers. However, Li [36] has assumed that x_1 and x_2 are real numbers which is certainly the simplest assumption but it does not allow capturing the very essence of the x_1 and x_2.

Hence, the mathematical formulation, i.e. problem P1, of such matrix games in which the payoffs are represented by trapezoidal intuitionistic fuzzy numbers, is not valid in general.

6. The Proposed Method

In this section, a new method, called the Mehar method, is proposed to find the optimal solution of problem P13 which yields the exact mathematical formulation of such matrix games in which payoffs are represented by trapezoidal intuitionistic fuzzy numbers.

The steps of the proposed Mehar method are as follows:

Problem P13

Maximize $(\tilde{v})$
Subject to:

$$\sum_{i=1}^{m}\tilde{a}_{ij}\tilde{x}_i \,\tilde{\geq}\, \tilde{v}, \qquad j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\tilde{x}_i \,\tilde{\cong}\, \tilde{1};$$

$$\tilde{x}_i \,\tilde{\geq}\, 0, \qquad i=1,2,\ldots,m.$$

Step 1: Since $\tilde{a}_{ij}$ are known, then by assuming

$\tilde{a}_{ij}=\left\langle\left(\underline{a}_{ij},a_{ij}^L,a_{ij}^U,\overline{a}_{ij}\right);w_{\tilde{a}ij},u_{\tilde{a}ij}\right\rangle,\tilde{x}_i=\left\langle\left(\underline{x}_i,x_i^L,x_i^U,\overline{x}_i\right);w,u\right\rangle,\tilde{v}=\left\langle\left(\underline{v},v^L,v^U,\overline{v}\right);w,u\right\rangle$, $\tilde{1}=\left\langle(1,1,1,1);w,u\right\rangle$ and $\tilde{0}=\left\langle(0,0,0,0);w,u\right\rangle$, where $w=\min\limits_{\substack{1\le i\le m\\1\le j\le n}}\left\{w_{\tilde{a}ij}\right\}$ and $u=\max\limits_{\substack{1\le i\le m\\1\le j\le n}}\left\{u_{\tilde{a}ij}\right\}$, problem P13 can be transformed into:

Problem P14

Maximize $\left(\left\langle\left(\underline{v},v^L,v^U,\overline{v}\right);w,u\right\rangle\right)$
Subject to:

$$\sum_{i=1}^{m}\left(\left\langle\left(\underline{a}_{ij},a_{ij}^L,a_{ij}^U,\overline{a}_{ij}\right);w_{\tilde{a}ij},u_{\tilde{a}ij}\right\rangle\right)\left(\left\langle\left(\underline{x}_i,x_i^L,x_i^U,\overline{x}_i\right);w,u\right\rangle\right)\tilde{\geq}\left(\left\langle\left(\underline{v},v^L,v^U,\overline{v}\right);w,u\right\rangle\right), j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\left\langle\left(\underline{x}_i,x_i^L,x_i^U,\overline{x}_i\right);w,u\right\rangle\right)\tilde{\cong}\left(\left\langle(1,1,1,1);w,u\right\rangle\right);$$

$$\left(\left\langle\left(\underline{x}_i,x_i^L,x_i^U,\overline{x}_i\right);w,u\right\rangle\right)\tilde{\geq}\left(\left\langle(0,0,0,0);w,u\right\rangle\right), \quad i=1,2,\ldots,m.$$

Step 2: Using the multiplication

$$\left\langle\left(\underline{a},a^L,a^U,\overline{a}\right);w_{\tilde{a}},u_{\tilde{a}}\right\rangle\left\langle\left(\underline{b},b^L,b^U,\overline{b}\right);w_{\tilde{b}},u_{\tilde{b}}\right\rangle$$
$$=\begin{cases}\left\langle\left(\underline{a}\,\underline{b},a^Lb^L,a^Ub^U,\overline{a}\,\overline{b}\right);\min\left(w_{\tilde{a}},w_{\tilde{b}}\right),\max\left(u_{\tilde{a}},u_{\tilde{b}}\right)\right\rangle, & \underline{a}\ge 0,\underline{b}\ge 0,\\ \left\langle\left(\underline{a}\overline{b},a^Lb^L,a^Ub^U,\overline{a}\,\overline{b}\right);\min\left(w_{\tilde{a}},w_{\tilde{b}}\right),\max\left(u_{\tilde{a}},u_{\tilde{b}}\right)\right\rangle, & \underline{a}<0,a^L\ge 0,\underline{b}\ge 0,\\ \left\langle\left(\underline{a}\overline{b},a^Lb^U,a^Ub^U,\overline{a}\,\overline{b}\right);\min\left(w_{\tilde{a}},w_{\tilde{b}}\right),\max\left(u_{\tilde{a}},u_{\tilde{b}}\right)\right\rangle, & a^L<0,a^U\ge 0,\underline{b}\ge 0,\\ \left\langle\left(\underline{a}\overline{b},a^Lb^U,a^Ub^L,\overline{a}\,\overline{b}\right);\min\left(w_{\tilde{a}},w_{\tilde{b}}\right),\max\left(u_{\tilde{a}},u_{\tilde{b}}\right)\right\rangle, & a^U<0,\overline{a}\ge 0,\underline{b}\ge 0,\\ \left\langle\left(\underline{a}\overline{b},a^Lb^U,a^Ub^L,\overline{a}\,\underline{b}\right);\min\left(w_{\tilde{a}},w_{\tilde{b}}\right),\max\left(u_{\tilde{a}},u_{\tilde{b}}\right)\right\rangle, & \overline{a}<0,\underline{b}\ge 0.\end{cases}$$

problem P14 can be transformed into:

Problem P15

Maximize $\left(\left\langle\left(\underline{v},v^L,v^U,\overline{v}\right);w,u\right\rangle\right)$
Subject to:

$$\sum_{i=1}^{m}\left(\left\langle\left(\underline{p}_{ij},p_{ij}^L,p_{ij}^U,\overline{p}_{ij}\right);w,u\right\rangle\right)\tilde{\geq}\left(\left\langle\left(\underline{v},v^L,v^U,\overline{v}\right);w,u\right\rangle\right),\ j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}\left(\left\langle\left(\underline{x}_i,x_i^L,x_i^U,\overline{x}_i\right);w,u\right\rangle\right)\tilde{=}\left(\left\langle(1,1,1,1);w,u\right\rangle\right);$$

$$\left(\left\langle\left(\underline{x}_i,x_i^L,x_i^U,\overline{x}_i\right);w,u\right\rangle\right)\tilde{\geq}\left(\left\langle(0,0,0,0),w,u\right\rangle\right),\quad i=1,2,\ldots,m.$$

where,
$$\left\langle\left(\underline{p}_{ij},p_{ij}^L,p_{ij}^U,\overline{p}_{ij}\right);w,u\right\rangle=\begin{cases}\left\langle\left(\underline{a}_{ij}\underline{x}_i,a_{ij}^Lx_i^L,a_{ij}^Ux_i^U,\overline{a}_{ij}\overline{x}_i\right);w,u\right\rangle, & \underline{a}_{ij}\geq 0,\\ \left\langle\left(\underline{a}_{ij}\overline{x}_i,a_{ij}^Lx_i^L,a_{ij}^Ux_i^U,\overline{a}_{ij}\overline{x}_i\right);w,u\right\rangle, & \underline{a}_{ij}<0,a_{ij}^L\geq 0,\\ \left\langle\left(\underline{a}_{ij}\overline{x}_i,a_{ij}^Lx_i^U,a_{ij}^Ux_i^U,\overline{a}_{ij}\overline{x}_i\right);w,u\right\rangle, & a_{ij}<0,a_{ij}^L<0,a_{ij}^U\geq 0,\\ \left\langle\left(\underline{a}_{ij}\overline{x}_i,a_{ij}^Lx_i^U,a_{ij}^Ux_i^L,\overline{a}_{ij}\overline{x}_i\right);w,u\right\rangle, & a_{ij}<0,a_{ij}^L<0,a_{ij}^U<0,\overline{a}_{ij}\geq 0,\\ \left\langle\left(\underline{a}_{ij}\overline{x}_i,a_{ij}^Lx_i^U,a_{ij}^Ux_i^L,\overline{a}_{ij}\underline{x}_i\right);w,u\right\rangle, & a_{ij}<0,a_{ij}^L<0,a_{ij}^U<0,\overline{a}_{ij}<0.\end{cases}$$

Step 3: Since $\sum_{i=1}^{n}\left(\left\langle\left(\underline{a}_i,a_i^L,a_i^U,\overline{a}_i\right);w,u\right\rangle\right)=\left(\left\langle\left(\sum_{i=1}^{n}\underline{a}_i,\sum_{i=1}^{n}a_i^L,\sum_{i=1}^{n}a_i^U,\sum_{i=1}^{n}\overline{a}_i\right);w,u\right\rangle\right)$, problem P15 can be transformed into:

Problem P16

Maximize $\left(\left\langle\left(\underline{v},v^L,v^U,\overline{v}\right);w,u\right\rangle\right)$
Subject to:

$$\left(\left\langle\left(\sum_{i=1}^{m}\underline{p}_{ij},\sum_{i=1}^{m}p_{ij}^L,\sum_{i=1}^{m}p_{ij}^U,\sum_{i=1}^{m}\overline{p}_{ij}\right);w,u\right\rangle\right)\tilde{\geq}\left(\left\langle\left(\underline{v},v^L,v^U,\overline{v}\right);w,u\right\rangle\right),\ j=1,2,\ldots,n;$$

$$\left(\left\langle\left(\sum_{i=1}^{m}\underline{x}_i,\sum_{i=1}^{m}x_i^L,\sum_{i=1}^{m}x_i^U,\sum_{i=1}^{m}\overline{x}_i\right);w,u\right\rangle\right)\tilde{=}\left(\left\langle(1,1,1,1);w,u\right\rangle\right);$$

$$\left(\left\langle\left(\underline{x}_i,x_i^L,x_i^U,\overline{x}_i\right);w,u\right\rangle\right)\tilde{\geq}\left(\left\langle(0,0,0,0);w,u\right\rangle\right),\quad i=1,2,\ldots,m.$$

Step 4: Since $\tilde{a}\,\tilde{\geq}\,b\Rightarrow(\tilde{a})_\alpha\geq(\tilde{b})_\alpha$ and $(\tilde{a})^\beta\geq(\tilde{b})^\beta$, problem P16 can be transformed into:

Problem P17

Maximize $\left(\left\langle\left(\underline{v},v^L,v^U,\overline{v}\right);w,u\right\rangle\right)_\alpha$, Maximize $\left(\left\langle\left(\underline{v},v^L,v^U,\overline{v}\right);w,u\right\rangle\right)^\beta$
Subject to:

$$\left(\left\langle\left(\sum_{i=1}^{m}\underline{p}_{ij},\sum_{i=1}^{m}p_{ij}^L,\sum_{i=1}^{m}p_{ij}^U,\sum_{i=1}^{m}\overline{p}_{ij}\right);w,u\right\rangle\right)_\alpha\geq\left(\left\langle\left(\underline{v},v^L,v^U,\overline{v}\right);w,u\right\rangle\right)_\alpha,\ j=1,2,\ldots,n;$$

$$\left(\left\langle\left(\sum_{i=1}^{m}\underline{p}_{ij},\sum_{i=1}^{m}p_{ij}^L,\sum_{i=1}^{m}p_{ij}^U,\sum_{i=1}^{m}\overline{p}_{ij}\right);w,u\right\rangle\right)^\beta\geq\left(\left\langle\left(\underline{v},v^L,v^U,\overline{v}\right);w,u\right\rangle\right)^\beta,\ j=1,2,\ldots,n;$$

$$\left(\left\langle\left(\sum_{i=1}^{m}\underline{x}_i,\sum_{i=1}^{m}x_i^L,\sum_{i=1}^{m}x_i^U,\sum_{i=1}^{m}\overline{x}_i\right);w,u\right\rangle\right)_\alpha=\left(\left\langle(1,1,1,1);w,u\right\rangle\right)_\alpha;$$

$$\left(\left\langle\left(\sum_{i=1}^{m}\underline{x}_i,\sum_{i=1}^{m}x_i^L,\sum_{i=1}^{m}x_i^U,\sum_{i=1}^{m}\overline{x}_i\right);w,u\right\rangle\right)^\beta=\left(\left\langle(1,1,1,1);w,u\right\rangle\right)^\beta;$$

$$\left(\left\langle\left(\underline{x}_i,x_i^L,x_i^U,\overline{x}_i\right);w,u\right\rangle\right)_\alpha\geq\left(\left\langle(0,0,0,0);w,u\right\rangle\right)_\alpha,i=1,2,\ldots,m;$$

$$\left(\left\langle\left(\underline{x}_i,x_i^L,x_i^U,\overline{x}_i\right);w,u\right\rangle\right)^\beta\geq\left(\left\langle(0,0,0,0);w,u\right\rangle\right)^\beta,i=1,2,\ldots,m.$$

where, $\alpha \in [0,w], \beta \in [u,1]$.

Step 5: Using the values $\left(\left\langle\left(\underline{a}_i, a_i^L, a_i^U, \overline{a}_i\right); w_{\tilde{a}_i}, u_{\tilde{a}_i}\right\rangle\right)_\alpha = \left[\frac{\left(w_{\tilde{a}_i}-\alpha\right)\underline{a}_i+\alpha a_i^L}{w_{\tilde{a}_i}}, \frac{\left(w_{\tilde{a}_i}-\alpha\right)\overline{a}_i+\alpha a_i^U}{w_{\tilde{a}_i}}\right]$ and $\left(\left\langle\left(\underline{a}_i, a_i^L, a_i^U, \overline{a}_i\right); w_{\tilde{a}_i}, u_{\tilde{a}_i}\right\rangle\right)^\beta = \left[\frac{(1-\beta)a_i^L+\left(\beta-u_{\tilde{a}_i}\right)\underline{a}_i}{1-u_{\tilde{a}_i}}, \frac{(1-\beta)a_i^U+\left(\beta-u_{\tilde{a}_i}\right)\overline{a}_i}{1-u_{\tilde{a}_i}}\right]$, problem P17 can be transformed into:

Problem P18

Maximize $\left\{\left[\frac{(w-\alpha)\underline{v}+\alpha v^L}{w}, \frac{(w-\alpha)\overline{v}+\alpha v^U}{w}\right]\right\}$,

Maximize $\left\{\left[\frac{(1-\beta)v^L+(\beta-u)\underline{v}}{1-u}, \frac{(1-\beta)v^U+(\beta-u)\overline{v}}{1-u}\right]\right\}$

Subject to:

$$\left[\frac{(w-\alpha)\sum_{i=1}^m \underline{p}_{ij}+\alpha\sum_{i=1}^m p_{ij}^L}{w}, \frac{(w-\alpha)\sum_{i=1}^m \overline{p}_{ij}+\alpha\sum_{i=1}^m p_{ij}^U}{w}\right] \geq \left[\frac{(w-\alpha)\underline{v}+\alpha v^L}{w}, \frac{(w-\alpha)\overline{v}+\alpha v^U}{w}\right], j=1,2,\ldots,n;$$

$$\left[\frac{(1-\beta)\sum_{i=1}^m p_{ij}^L+(\beta-u)\sum_{i=1}^m \underline{p}_{ij}}{1-u}, \frac{(1-\beta)\sum_{i=1}^m p_{ij}^U+(\beta-u)\sum_{i=1}^m \overline{p}_{ij}}{1-u}\right] \geq \left[\frac{(1-\beta)v^L+(\beta-u)\underline{v}}{1-u}, \frac{(1-\beta)v^U+(\beta-u)\overline{v}}{1-u}\right], j=1,2,\ldots,n;$$

$$\left[\frac{(w-\alpha)\sum_{i=1}^m \underline{x}_i+\alpha\sum_{i=1}^m x_i^L}{w}, \frac{(w-\alpha)\sum_{i=1}^m \overline{x}_i+\alpha\sum_{i=1}^m x_i^U}{w}\right] = \left[\frac{(w-\alpha)+\alpha}{w}, \frac{(w-\alpha)+\alpha}{w}\right];$$

$$\left[\frac{(1-\beta)\sum_{i=1}^m x_i^L+(\beta-u)\sum_{i=1}^m \underline{x}_i}{1-u}, \frac{(1-\beta)\sum_{i=1}^m x_i^U+(\beta-u)\sum_{i=1}^m \overline{x}_i}{1-u}\right] = \left[\frac{(1-\beta)+(\beta-u)}{1-u}, \frac{(1-\beta)+(\beta-u)}{1-u}\right];$$

$$\left[\frac{(w-\alpha)\underline{x}_i+\alpha x_i^L}{w}, \frac{(w-\alpha)\overline{x}_i+\alpha x_i^U}{w}\right] \geq [0,0], i=1,2,\ldots,m;$$

$$\left[\frac{(1-\beta)x_i^L+(\beta-u)\underline{x}_i}{1-u}, \frac{(1-\beta)x_i^U+(\beta-u)\overline{x}_i}{1-u}\right] \geq [0,0], i=1,2,\ldots,m.$$

where, $\alpha \in [0,w], \beta \in [u,1]$.

Step 6: Using the property $[a,b] \geq [c,d] \Rightarrow a \geq c, b \geq d$, problem P18 can be transformed into:

Problem P19

Maximize $\left(\left[\frac{(w-\alpha)\underline{v}+\alpha v^L}{w}, \frac{(w-\alpha)\overline{v}+\alpha v^U}{w}\right]\right)$

Maximize $\left(\left[\frac{(1-\beta)v^L+(\beta-u)\underline{v}}{1-u}, \frac{(1-\beta)v^U+(\beta-u)\overline{v}}{1-u}\right]\right)$

Subject to:

$$\frac{(w-\alpha)\sum_{i=1}^{m}\underline{p}_{ij}+\alpha\sum_{i=1}^{m}p_{ij}^{L}}{w}\geq\frac{(w-\alpha)\underline{v}+\alpha v^{L}}{w},\qquad j=1,2,\ldots,n;$$

$$\frac{(w-\alpha)\sum_{i=1}^{m}\overline{p}_{ij}+\alpha\sum_{i=1}^{m}p_{ij}^{U}}{w}\geq\frac{(w-\alpha)\overline{v}+\alpha v^{U}}{w},\qquad j=1,2,\ldots,n;$$

$$\frac{(1-\beta)\sum_{i=1}^{m}p_{ij}^{L}+(\beta-u)\sum_{i=1}^{m}\underline{p}_{ij}}{1-u}\geq\frac{(1-\beta)v^{L}+(\beta-u)\underline{v}}{1-u},j=1,2,\ldots,n;$$

$$\frac{(1-\beta)\sum_{i=1}^{m}p_{ij}^{U}+(\beta-u)\sum_{i=1}^{m}\overline{p}_{ij}}{1-u}\geq\frac{(1-\beta)v^{U}+(\beta-u)\overline{v}}{1-u},j=1,2,\ldots,n;$$

$$\frac{(w-\alpha)\sum_{i=1}^{m}\underline{x}_{i}+\alpha\sum_{i=1}^{m}x_{i}^{L}}{w}=\frac{(w-\alpha)+\alpha}{w};$$

$$\frac{(w-\alpha)\sum_{i=1}^{m}\overline{x}_{i}+\alpha\sum_{i=1}^{m}x_{i}^{U}}{w}=\frac{(w-\alpha)+\alpha}{w};$$

$$\frac{(1-\beta)\sum_{i=1}^{m}x_{i}^{L}+(\beta-u)\sum_{i=1}^{m}\underline{x}_{i}}{1-u}=\frac{(1-\beta)+(\beta-u)}{1-u};$$

$$\frac{(1-\beta)\sum_{i=1}^{m}x_{i}^{U}+(\beta-u)\sum_{i=1}^{m}\overline{x}_{i}}{1-u}=\frac{(1-\beta)+(\beta-u)}{1-u};$$

$$\frac{(w-\alpha)\overline{v}+\alpha v^{U}}{w}-\frac{(w-\alpha)\underline{v}+\alpha v^{L}}{w}\geq 0;$$

$$\frac{(1-\beta)v^{U}+(\beta-u)\overline{v}}{1-u}-\frac{(1-\beta)v^{L}+(\beta-u)\underline{v}}{1-u}\geq 0;$$

$$\frac{(w-\alpha)\underline{x}_{i}+\alpha x_{i}^{L}}{w}\geq 0,\qquad i=1,2,\ldots,m;$$

$$\frac{(w-\alpha)\overline{x}_{i}+\alpha x_{i}^{U}}{w}\geq 0,\qquad i=1,2,\ldots,m;$$

$$\frac{(w-\alpha)\overline{x}_{i}+\alpha x_{i}^{U}}{w}-\frac{(w-\alpha)\underline{x}_{i}+\alpha x_{i}^{L}}{w}\geq 0,i=1,2,\ldots,m;$$

$$\left(\frac{(1-\beta)x_{i}^{L}+(\beta-u)\underline{x}_{i}}{1-u}\right)\geq 0,\qquad i=1,2,\ldots,m;$$

$$\left(\frac{(1-\beta)x_{i}^{U}+(\beta-u)\overline{x}_{i}}{1-u}\right)\geq 0,\qquad i=1,2,\ldots,m;$$

$$\left(\frac{(1-\beta)x_{i}^{U}+(\beta-u)\overline{x}_{i}}{1-u}\right)-\left(\frac{(1-\beta)x_{i}^{L}+(\beta-u)\underline{x}_{i}}{1-u}\right)\geq 0,\ i=1,2,\ldots,m.$$

where, $\alpha\in[0,w],\beta\in[u,1]$.

Step 7: The two interval-valued objective functions in problem P19 may be regarded to be of equal importance, i.e., with weights 0.5. Therefore, using the usual linear weighted average (cf. [20, 25, 28]), problem P19 can be transformed into the following interval- valued mathematical programming problem:

Problem P20

$$\text{Maximize}\quad \left(\left[\frac{\left(\frac{(w-\alpha)\underline{v}+\alpha v^{L}}{w}\right)+\left(\frac{(1-\beta)v^{L}+(\beta-u)\underline{v}}{1-u}\right)}{2},\frac{\left(\frac{(w-\alpha)\overline{v}+\alpha v^{U}}{w}\right)+\left(\frac{(1-\beta)v^{U}+(\beta-u)\overline{v}}{1-u}\right)}{2}\right]\right)$$

Subject to:
Constraints of problem P19.

Step 8: According to Ishibushi and Tanaka [26], the interval-valued objective function $[a,b]$ is equivalent to the biobjective objective function $\left[a,\frac{a+b}{2}\right]$, and therefore problem P20 can be transformed into:

Problem P21

$$\text{Maximize}\quad \left(\left[\frac{\left(\frac{(w-\alpha)\underline{v}+\alpha v^{L}}{w}\right)+\left(\frac{(1-\beta)v^{L}+(\beta-u)\underline{v}}{1-u}\right)}{2},\ \frac{\left(\frac{(w-\alpha)\underline{v}+\alpha v^{L}}{w}\right)+\left(\frac{(1-\beta)v^{L}+(\beta-u)\underline{v}}{1-u}\right)+\left(\frac{(w-\alpha)\overline{v}+\alpha v^{U}}{w}\right)+\left(\frac{(1-\beta)v^{U}+(\beta-u)\overline{v}}{1-u}\right)}{4}\right]\right)$$

Subject to
Constraints of problem P19.

Step 9: Using the usual linear weighted average (cf. [20, 25, 28]), problem P21 can be transformed into:

Problem P22

$$\text{Maximize}\quad \left(\frac{1}{2}\left(\frac{\left(\frac{(w-\alpha)\underline{v}+\alpha v^{L}}{w}\right)+\left(\frac{(1-\beta)v^{L}+(\beta-u)\underline{v}}{1-u}\right)}{2}\right)+\frac{1}{2}\left(\frac{\left(\frac{(w-\alpha)\underline{v}+\alpha v^{L}}{w}\right)+\left(\frac{(1-\beta)v^{L}+(\beta-u)\underline{v}}{1-u}\right)+\left(\frac{(w-\alpha)\overline{v}+\alpha v^{U}}{w}\right)+\left(\frac{(1-\beta)v^{U}+(\beta-u)\overline{v}}{1-u}\right)}{4}\right)\right)$$

Subject to:
Constraints of problem P19.

Step 10: Find the optimal solution $\{\underline{x}_i=e_i,\overline{x}_i=f_i,\underline{v}=q,\overline{v}=r\}$ of problem P22 by taking α=0, β=1.

Step 11: Find the optimal solution $\{x_i^{L}=g_i,x_i^{U}=h_i,v^{L}=s,v^{U}=t\}$ of the following problem by taking α=*w* and β=*u*:

Problem P23

$$\text{Maximize}\quad \left(\frac{1}{2}\left(\frac{\left(\frac{(w-\alpha)\underline{v}+\alpha v^{L}}{w}\right)+\left(\frac{(1-\beta)v^{L}+(\beta-u)\underline{v}}{1-u}\right)}{2}\right)+\frac{1}{2}\left(\frac{\left(\frac{(w-\alpha)\underline{v}+\alpha v^{L}}{w}\right)+\left(\frac{(1-\beta)v^{L}+(\beta-u)\underline{v}}{1-u}\right)+\left(\frac{(w-\alpha)\overline{v}+\alpha v^{U}}{w}\right)+\left(\frac{(1-\beta)v^{U}+(\beta-u)\overline{v}}{1-u}\right)}{4}\right)\right)$$

Subject to:
Constraints of problem P19 with the following additional constraints:

$x_i^L \geq e_i, \quad i = 1,2,\dots,m;$
$x_i^U \leq f_i, \quad i = 1,2,\dots,m;$
$v^L \geq q;$
$v^U \leq r.$

Step 12: Using the optimal solution obtained in Steps 10 and 11, the fuzzy optimal solution of problem P13 is

$$\left\{\tilde{v} = \left\langle\left(\underline{v}, v^L, v^U, \bar{v}\right); w, u\right\rangle, \tilde{x}_i = \left\langle\left(\underline{x}_i, x_i^L, x_i^U, \bar{x}_i\right); w, u\right\rangle; i = 1,2,\dots,m\right\}.$$

7. Numerical example

Li [36] solved problem P12 to illustrate his proposed method. However, as discussed in Section 5, the correct mathematical formulation of the problem, as chosen by Li [36], is problem P24.

In this section, to illustrate the proposed Mehar method, the exact optimal solution of the following problem P24 is obtained:

Problem P24

Maximize $(\tilde{v})$
Subject to:

$$\left\langle(175,180,190);0.6,0.2\right\rangle\tilde{x}_1 + \left\langle(80,90,100);0.9,0.1\right\rangle\tilde{x}_2 \tilde{\geq} \tilde{v};$$
$$\left\langle(150,156,158);0.6,0.1\right\rangle\tilde{x}_1 + \left\langle(175,180,190);0.6,0.2\right\rangle\tilde{x}_2 \tilde{\geq} \tilde{v};$$

$$\tilde{x}_1 + \tilde{x}_2 \tilde{=} \tilde{1};$$
$$\tilde{x}_1, \tilde{x}_2 \tilde{\geq} \tilde{0}.$$

The exact fuzzy optimal solution of problem P24 can be obtained as follows:

Step 1: Since $\tilde{a}_{ij}$ are known, then by assuming $\tilde{x}_i = \left\langle(\underline{x}_i, x_i, \bar{x}_i);0.6,0.2\right\rangle, \tilde{v} = \left\langle(\underline{v}, v, \bar{v});0.6,0.2\right\rangle$, $\tilde{1} = \left\langle(1,1,1);0.6,0.2\right\rangle$ and $\tilde{0} = \left\langle(0,0,0);0.6,0.2\right\rangle$, where $0.6 = \min\limits_{\substack{1\leq i\leq 2\\1\leq j\leq 2}}\left\{w_{\tilde{a}_{ij}}\right\}$ and $0.2 = \max\limits_{\substack{1\leq i\leq 2\\1\leq j\leq 2}}\left\{u_{\tilde{a}_{ij}}\right\}$, problem P24 can be transformed into:

Problem P25

Maximize $\left(\left\langle(\underline{v}, v, \bar{v});0.6,0.2\right\rangle\right)$
Subject to:

$$\left\langle(175,180,190);0.6,0.2\right\rangle\left\langle(\underline{x}_1, x_1, \bar{x}_1);0.6,0.2\right\rangle + \left\langle(80,90,100);0.9,0.1\right\rangle\left\langle(\underline{x}_2, x_2, \bar{x}_2);0.6,0.2\right\rangle \tilde{\geq} \left\langle(\underline{v}, v, \bar{v});0.6,0.2\right\rangle;$$
$$\left\langle(150,156,158);0.6,0.1\right\rangle\left\langle(\underline{x}_1, x_1, \bar{x}_1);0.6,0.2\right\rangle + \left\langle(175,180,190);0.6,0.2\right\rangle\left\langle(\underline{x}_2, x_2, \bar{x}_2);0.6,0.2\right\rangle \tilde{\geq} \left\langle(\underline{v}, v, \bar{v});0.6,0.2\right\rangle;$$
$$\left\langle(\underline{x}_1, x_1, \bar{x}_1);0.6,0.2\right\rangle + \left\langle(\underline{x}_2, x_2, \bar{x}_2);0.6,0.2\right\rangle \tilde{=} \left\langle(1,1,1);0.6,0.2\right\rangle;$$
$$\left\langle(\underline{x}_1, x_1, \bar{x}_1);0.6,0.2\right\rangle \tilde{\geq} \left\langle(0,0,0);0.6,0.2\right\rangle; \left\langle(\underline{x}_2, x_2, \bar{x}_2);0.6,0.2\right\rangle \tilde{\geq} \left\langle(0,0,0);0.6,0.2\right\rangle.$$

Step 2: Problem P25 can be transformed into:

Problem P26

Maximize $\left(\left\langle(\underline{v}, v, \bar{v});0.6,0.2\right\rangle\right)$
Subject to:

$$\left\langle(175\underline{x}_1, 180x_1, 190\bar{x}_1);0.6,0.2\right\rangle + \left\langle(80\underline{x}_2, 90x_2, 100\bar{x}_2);0.6,0.2\right\rangle \tilde{\geq} \left\langle(\underline{v}, v, \bar{v});0.6,0.2\right\rangle;$$
$$\left\langle(150\underline{x}_1, 156x_1, 158\bar{x}_1);0.6,0.2\right\rangle + \left\langle(175\underline{x}_2, 180x_2, 190\bar{x}_2);0.6,0.2\right\rangle \tilde{\geq} \left\langle(\underline{v}, v, \bar{v});0.6,0.2\right\rangle;$$

$\langle(\underline{x}_1, x_1, \overline{x}_1); 0.6, 0.2\rangle + \langle(\underline{x}_2, x_2, \overline{x}_2); 0.6, 0.2\rangle \tilde{=} \langle(1,1,1); 0.6, 0.2\rangle$;
$\langle(\underline{x}_1, x_1, \overline{x}_1); 0.6, 0.2\rangle \tilde{\geq} \langle(0,0,0); 0.6, 0.2\rangle; \langle(\underline{x}_2, x_2, \overline{x}_2); 0.6, 0.2\rangle \tilde{\geq} \langle(0,0,0); 0.6, 0.2\rangle$.

Step 3: Since $\sum_{i=1}^{n}\left(\langle(\underline{a}_i, a_i, \overline{a}_i); w, u\rangle\right) = \left(\left\langle\left(\sum_{i=1}^{n}\underline{a}_i, \sum_{i=1}^{n}a_i, \sum_{i=1}^{n}\overline{a}_i\right); w, u\right\rangle\right)$, problem P26 can be transformed into:

Problem P27

Maximize $\left(\langle(\underline{v}, v, \overline{v}); 0.6, 0.2\rangle\right)$
Subject to:

$\langle(175\underline{x}_1 + 80\underline{x}_2, 180x_1 + 90x_2, 190\overline{x}_1 + 100\overline{x}_2); 0.6, 0.2\rangle \tilde{\geq} \langle(\underline{v}, v, \overline{v}); 0.6, 0.2\rangle$;
$\langle(150\underline{x}_1 + 175\underline{x}_2, 156x_1 + 180x_2, 158\overline{x}_1 + 190\overline{x}_2); 0.6, 0.2\rangle \tilde{\geq} \langle(\underline{v}, v, \overline{v}); 0.6, 0.2\rangle$;
$\langle(\underline{x}_1 + \underline{x}_2, x_1 + x_2, \overline{x}_1 + \overline{x}_2); 0.6, 0.2\rangle \tilde{=} \langle(1,1,1); 0.6, 0.2\rangle$;
$\langle(\underline{x}_1, x_1, \overline{x}_1); 0.6, 0.2\rangle \tilde{\geq} \langle(0,0,0); 0.6, 0.2\rangle; \langle(\underline{x}_2, x_2, \overline{x}_2); 0.6, 0.2\rangle \tilde{\geq} \langle(0,0,0); 0.6, 0.2\rangle$.

Step 4: Using the properties $\tilde{a} \tilde{\geq} \tilde{b} \Rightarrow (\tilde{a})_\alpha \geq (\tilde{b})_\alpha$ and $(\tilde{a})^\beta \geq (\tilde{b})^\beta$, problem P27 can be transformed into:

Problem P28

Maximize $\left(\langle(\underline{v}, v, \overline{v}); 0.6, 0.2\rangle\right)_\alpha$, Maximize $\left(\langle(\underline{v}, v, \overline{v}); 0.6, 0.2\rangle\right)^\beta$

Subject to:

$\left(\langle(175\underline{x}_1 + 80\underline{x}_2, 180x_1 + 90x_2, 190\overline{x}_1 + 100\overline{x}_2); 0.6, 0.2\rangle\right)_\alpha \geq \left(\langle(\underline{v}, v, \overline{v}); 0.6, 0.2\rangle\right)_\alpha$;
$\left(\langle(175\underline{x}_1 + 80\underline{x}_2, 180x_1 + 90x_2, 190\overline{x}_1 + 100\overline{x}_2); 0.6, 0.2\rangle\right)^\beta \geq \left(\langle(\underline{v}, v, \overline{v}); 0.6, 0.2\rangle\right)^\beta$;
$\left(\langle(150\underline{x}_1 + 175\underline{x}_2, 156x_1 + 180x_2, 158\overline{x}_1 + 190\overline{x}_2); 0.6, 0.2\rangle\right)_\alpha \geq \left(\langle(\underline{v}, v, \overline{v}); 0.6, 0.2\rangle\right)_\alpha$;
$\left(\langle(150\underline{x}_1 + 175\underline{x}_2, 156x_1 + 180x_2, 158\overline{x}_1 + 190\overline{x}_2); 0.6, 0.2\rangle\right)^\beta \geq \left(\langle(\underline{v}, v, \overline{v}); 0.6, 0.2\rangle\right)^\beta$;
$\left(\langle(\underline{x}_1 + \underline{x}_2, x_1 + x_2, \overline{x}_1 + \overline{x}_2); 0.6, 0.2\rangle\right)_\alpha = \left(\langle(1,1,1); 0.6, 0.2\rangle\right)_\alpha$;
$\left(\langle(\underline{x}_1 + \underline{x}_2, x_1 + x_2, \overline{x}_1 + \overline{x}_2); 0.6, 0.2\rangle\right)^\beta = \left(\langle(1,1,1); 0.6, 0.2\rangle\right)^\beta$;
$\left(\langle(\underline{x}_1, x_1, \overline{x}_1); 0.6, 0.2\rangle\right)_\alpha \geq \left(\langle(0,0,0); 0.6, 0.2\rangle\right)_\alpha$;
$\left(\langle(\underline{x}_1, x_1, \overline{x}_1); 0.6, 0.2\rangle\right)^\beta \geq \left(\langle(0,0,0); 0.6, 0.2\rangle\right)^\beta$;
$\left(\langle(\underline{x}_1, x_1, \overline{x}_1); 0.6, 0.2\rangle\right)_\alpha \geq \left(\langle(0,0,0); 0.6, 0.2\rangle\right)_\alpha$;
$\left(\langle(\underline{x}_1, x_1, \overline{x}_1); 0.6, 0.2\rangle\right)^\beta \geq \left(\langle(0,0,0); 0.6, 0.2\rangle\right)^\beta$;
$\left(\langle(\underline{x}_2, x_2, \overline{x}_2); 0.6, 0.2\rangle\right)_\alpha \geq \left(\langle(0,0,0); 0.6, 0.2\rangle\right)_\alpha$;
$\left(\langle(\underline{x}_2, x_2, \overline{x}_2); 0.6, 0.2\rangle\right)^\beta \geq \left(\langle(0,0,0); 0.6, 0.2\rangle\right)^\beta$.

where, $\alpha \in [0, 0.6]$ and $\beta \in [0.2, 1]$.

Step 5: Using the values $\left(\langle(\underline{a}_i, a_i^L, a_i^U, \overline{a}_i); w_{\tilde{a}_i}, u_{\tilde{a}_i}\rangle\right)_\alpha = \left[\frac{(w_{\tilde{a}_i} - \alpha)\underline{a}_i + \alpha a_i}{w_{\tilde{a}_i}}, \frac{(w_{\tilde{a}_i} - \alpha)\overline{a}_i + \alpha a_i}{w_{\tilde{a}_i}}\right]$ and $\left(\langle(\underline{a}_i, a_i^L, a_i^U, \overline{a}_i); w_{\tilde{a}_i}, u_{\tilde{a}_i}\rangle\right)^\beta =$
$\left[\frac{(1-\beta)a_i + (\beta - u_{\tilde{a}_i})\underline{a}_i}{1 - u_{\tilde{a}_i}}, \frac{(1-\beta)a_i + (\beta - u_{\tilde{a}_i})\overline{a}_i}{1 - u_{\tilde{a}_i}}\right]$, problem P28 can be transformed into:

Problem P29

$$\text{Maximize}\left[\frac{(0.6-\alpha)\underline{v}+\alpha v}{0.6},\frac{(0.6-\alpha)\overline{v}+\alpha v}{0.6}\right]$$

$$\text{Maximize}\left[\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2},\frac{(1-\beta)v+(\beta-0.2)\overline{v}}{1-0.2}\right]$$

Subject to:

$$\left[\frac{(0.6-\alpha)(175\underline{x}_1+80\underline{x}_2)+\alpha(180x_1+90x_2)}{0.6},\frac{(0.6-\alpha)(190\overline{x}_1+100\overline{x}_2)+\alpha(180x_1+90x_2)}{0.6}\right]\geq \left[\frac{(0.6-\alpha)\underline{v}+\alpha v}{0.6},\frac{(0.6-\alpha)\overline{v}+\alpha v}{0.2}\right];$$

$$\left[\frac{(1-\beta)(180x_1+90x_2)+(\beta-0.2)(175\underline{x}_1+80\underline{x}_2)}{1-0.2},\frac{(1-\beta)(180x_1+90x_2)+(\beta-0.2)(190\overline{x}_1+100\overline{x}_2)}{1-0.2}\right]\geq \left[\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2},\frac{(1-\beta)v+(\beta-0.2)\overline{v}}{1-0.2}\right];$$

$$\left[\frac{(0.6-\alpha)(150\underline{x}_1+175\underline{x}_2)+\alpha(156x_1+180x_2)}{0.6},\frac{(0.6-\alpha)(158\overline{x}_1+190\overline{x}_2)+\alpha(156x_1+180x_2)}{0.6}\right]\geq \left[\frac{(0.6-\alpha)\underline{v}+\alpha v}{0.6},\frac{(0.6-\alpha)\overline{v}+\alpha v}{0.6}\right];$$

$$\left[\frac{(1-\beta)(156x_1+180x_2)+(\beta-0.2)(150\underline{x}_1+175\underline{x}_2)}{1-0.2},\frac{(1-\beta)(156x_1+180x_2)+(\beta-0.2)(158\overline{x}_1+190\overline{x}_2)}{1-0.2}\right]\geq \left[\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2},\frac{(1-\beta)v+(\beta-0.2)\overline{v}}{1-0.2}\right];$$

$$\left[\frac{(0.6-\alpha)(\underline{x}_1+\underline{x}_2)+\alpha(x_1+x_2)}{0.6},\frac{(0.6-\alpha)(\overline{x}_1+\overline{x}_2)+\alpha(x_1+x_2)}{0.6}\right]=\left[\frac{(0.6-\alpha)+\alpha}{0.6},\frac{(0.6-\alpha)+\alpha}{0.6}\right];$$

$$\left[\frac{(1-\beta)(x_1+x_2)+(\beta-0.2)(\underline{x}_1+\underline{x}_2)}{1-0.2},\frac{(1-\beta)(x_1+x_2)+(\beta-0.2)(\overline{x}_1+\overline{x}_2)}{1-0.2}\right]=\left[\frac{(1-\beta)+(\beta-0.2)}{1-0.2},\frac{(1-\beta)+(\beta-0.2)}{1-0.2}\right];$$

$$\left[\frac{(0.6-\alpha)\underline{x}_1+\alpha x_1}{0.6},\frac{(0.6-\alpha)\overline{x}_1+\alpha x_1}{0.6}\right]\geq[0,0];$$

$$\left[\frac{(1-\beta)x_1+(\beta-0.2)\underline{x}_1}{1-0.2},\frac{(1-\beta)x_1+(\beta-0.2)\overline{x}_1}{1-0.2}\right]\geq[0,0];$$

$$\left[\frac{(0.6-\alpha)\underline{x}_2+\alpha x_2}{0.6},\frac{(0.6-\alpha)\overline{x}_2+\alpha x_2}{0.6}\right]\geq[0,0];$$

$$\left[\frac{(1-\beta)x_2+(\beta-0.2)\underline{x}_2}{1-0.2},\frac{(1-\beta)x_2+(\beta-0.2)\overline{x}_2}{1-0.2}\right]\geq[0,0].$$

where: $\alpha\in[0,0.6]$ and $\beta\in[0.2,1]$.

Step 6: Using the property $[a,b]\geq[c,d]\Rightarrow a\geq c, b\geq d$, problem P29 can be transformed into:

Problem P30

$$\text{Maximize}\left\{\left[\frac{(0.6-\alpha)\underline{v}+\alpha v}{0.6},\frac{(0.6-\alpha)\overline{v}+\alpha v}{0.6}\right]\right\}$$

$$\text{Maximize}\left\{\left[\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2},\frac{(1-\beta)v+(\beta-0.2)\overline{v}}{1-0.2}\right]\right\}$$

Subject to

$$\frac{(0.6-\alpha)(175\underline{x}_1+80\underline{x}_2)+\alpha(180x_1+90x_2)}{0.6}\geq\frac{(0.6-\alpha)\underline{v}+\alpha v}{0.6};$$

$$\frac{(0.6-\alpha)(190\overline{x}_1+100\overline{x}_2)+\alpha(180x_1+90x_2)}{0.6}\geq\frac{(0.6-\alpha)\overline{v}+\alpha v}{0.6};$$

$$\frac{(1-\beta)(180x_1+90x_2)+(\beta-0.2)(175\underline{x}_1+80\underline{x}_2)}{1-0.2}\geq\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2};$$

$$\frac{(1-\beta)(180x_1+90x_2)+(\beta-0.2)(190\overline{x}_1+100\overline{x}_2)}{1-0.2}\geq\frac{(1-\beta)v+(\beta-0.2)\overline{v}}{1-0.2};$$

$$\frac{(0.6-\alpha)(150\underline{x}_1+175\underline{x}_2)+\alpha(156x_1+180x_2)}{0.6}\geq\frac{(0.6-\alpha)\underline{v}+\alpha v}{w};$$

$$\frac{(0.6-\alpha)(158\overline{x}_1+190\overline{x}_2)+\alpha(156x_1+180x_2)}{0.6}\geq\frac{(0.6-\alpha)\overline{v}+\alpha v}{w};$$

$$\frac{(1-\beta)(156x_1+180x_2)+(\beta-0.2)(150\underline{x}_1+175\underline{x}_2)}{1-0.2}\geq\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2};$$

$$\frac{(1-\beta)(156x_1+180x_2)+(\beta-0.2)(158\overline{x}_1+190\overline{x}_2)}{1-0.2}\geq\frac{(1-\beta)v+(\beta-0.2)\overline{v}}{1-0.2};$$

$$\frac{(0.6-\alpha)(\underline{x}_1+\underline{x}_2)+\alpha(x_1+x_2)}{0.6}=\frac{(0.6-\alpha)+\alpha}{0.6};$$

$$\frac{(0.6-\alpha)(\overline{x}_1+\overline{x}_2)+\alpha(x_1+x_2)}{0.6}=\frac{(0.6-\alpha)+\alpha}{0.6};$$

$$\frac{(1-\beta)(x_1+x_2)+(\beta-0.2)(\underline{x}_1+\underline{x}_2)}{1-0.2}=\frac{(1-\beta)+(\beta-0.2)}{1-0.2};$$

$$\frac{(1-\beta)(x_1+x_2)+(\beta-u)(\overline{x}_1+\overline{x}_2)}{1-0.2}=\frac{(1-\beta)+(\beta-0.2)}{1-0.2};$$

$$\frac{(0.6-\alpha)\overline{v}+\alpha v}{0.6}-\frac{(0.6-\alpha)\underline{v}+\alpha v}{0.6}\geq 0;$$

$$\frac{(1-\beta)v+(\beta-0.2)\overline{v}}{1-0.2}-\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2}\geq 0;$$

$$\frac{(0.6-\alpha)\underline{x}_1+\alpha x_1}{0.6}\geq 0;$$

$$\frac{(0.6-\alpha)\overline{x}_1+\alpha x_1}{0.6}\geq 0;$$

$$\frac{(0.6-\alpha)\overline{x}_1+\alpha x_1}{0.6}-\frac{(0.6-\alpha)\underline{x}_1+\alpha x_1}{0.6}\geq 0;$$

$$\frac{(1-\beta)x_1+(\beta-0.2)\underline{x}_1}{1-0.2}\geq 0;$$

$$\frac{(1-\beta)x_1+(\beta-0.2)\overline{x}_1}{1-0.2}\geq 0;$$

$$\frac{(1-\beta)x_1+(\beta-0.2)\overline{x}_1}{1-0.2}-\frac{(1-\beta)x_1+(\beta-0.2)\underline{x}_1}{1-0.2}\geq 0;$$

$$\frac{(0.6-\alpha)\underline{x}_2+\alpha x_2}{0.6}\geq 0;$$

$$\frac{(0.6-\alpha)\overline{x}_2+\alpha x_2}{0.6}\geq 0;$$

$$\frac{(0.6-\alpha)\overline{x}_2+\alpha x_2}{0.6}-\frac{(0.6-\alpha)\underline{x}_2+\alpha x_2}{0.6}\geq 0;$$

$$\frac{(1-\beta)x_2+(\beta-0.2)\underline{x}_2}{1-0.2}\geq 0;$$

$$\frac{(1-\beta)x_2+(\beta-0.2)\overline{x}_2}{1-0.2}\geq 0;$$
$$\frac{(1-\beta)x_2+(\beta-0.2)\overline{x}_2}{1-0.2}-\frac{(1-\beta)x_2+(\beta-0.2)\underline{x}_2}{1-0.2}\geq 0.$$

where: $\alpha\in[0,0.6]$ and $\beta\in[0.2,1]$.

Step 7: Using the linear weighted average, problem P30 can be transformed into the following interval- valued mathematical programming problem P31:

Problem P31

$$\text{Maximize}\left(\left[\frac{\left(\frac{(0.6-\alpha)\underline{v}+\alpha v}{0.6}\right)+\left(\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2}\right)}{2},\frac{\left(\frac{(0.6-\alpha)\overline{v}+\alpha v}{0.6}\right)+\left(\frac{(1-\beta)v+(\beta-0.2)\overline{v}}{1-0.2}\right)}{2}\right]\right)$$

Subject to
Constraints of problem P30.

Step 8: Problem P31 can be transformed into the following problem P32:

Problem P32

$$\text{Maximize}\left(\left[\begin{array}{l}\frac{\left(\frac{(0.6-\alpha)\underline{v}+\alpha v}{0.6}\right)+\left(\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2}\right)}{2},\\ \frac{\left(\frac{(0.6-\alpha)\underline{v}+\alpha v}{0.6}\right)+\left(\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2}\right)+\left(\frac{(0.6-\alpha)\overline{v}+\alpha v}{0.6}\right)+\left(\frac{(1-\beta)v+(\beta-0.2)\overline{v}}{1-0.2}\right)}{4}\end{array}\right]\right)$$

Subject to

Constraints of problem P30.

Step 9: Using the linear weighted average, problem P32 can be transformed into:

Problem P33

$$\text{Maximize}\left(\begin{array}{l}\frac{1}{2}\left(\frac{\left(\frac{(0.6-\alpha)\underline{v}+\alpha v}{0.6}\right)+\left(\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2}\right)}{2}\right)+\\ \frac{1}{2}\left(\frac{\left(\frac{(0.6-\alpha)\underline{v}+\alpha v}{0.6}\right)+\left(\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2}\right)+\left(\frac{(0.6-\alpha)\overline{v}+\alpha v}{0.6}\right)+\left(\frac{(1-\beta)v+(\beta-0.2)\overline{v}}{1-0.2}\right)}{4}\right)\end{array}\right)$$

Subject to:

Constraints of problem P30.

Step 10: The optimal solution of problem P33, by taking α=0 and β=1, is

$$\left\{\underline{v}=\frac{3725}{24},\overline{v}=\frac{494}{3},\overline{x}_1=\frac{19}{24},\underline{x}_1=\frac{19}{24},\overline{x}_2=\frac{19}{24},\underline{x}_2=\frac{19}{24}\right\}.$$

Step 11: By taking α=0.6 and β=1, the optimal solution of:

Problem P34

$$\text{Maximize}\left\{\frac{1}{2}\left(\frac{\left(\frac{(0.6-\alpha)\underline{v}+\alpha v}{0.6}\right)+\left(\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2}\right)}{2}\right)+\frac{1}{2}\left(\frac{\left(\frac{(0.6-\alpha)\underline{v}+\alpha v}{0.6}\right)+\left(\frac{(1-\beta)v+(\beta-0.2)\underline{v}}{1-0.2}\right)+\left(\frac{(0.6-\alpha)\overline{v}+\alpha v}{0.6}\right)+\left(\frac{(1-\beta)v+(\beta-0.2)\overline{v}}{1-0.2}\right)}{4}\right)\right\}$$

Subject to

Constraints of problem P30 with the following additional constraints

$$x_1 \geq \frac{19}{24};\quad x_1 \leq \frac{19}{24};\quad x_2 \geq \frac{5}{24};\quad x_2 \leq \frac{5}{24};\quad v \geq \frac{3725}{24};\quad v \leq \frac{494}{3};\quad v \geq \frac{3725}{24};\quad v \leq \frac{494}{3}.$$

is $\left\{x_1 = \frac{19}{24}, x_2 = \frac{5}{24}, v = 161\right\}$.

Step 12: Using the optimal solutions obtained in Step 10 and Step 11, the fuzzy optimal solution of problem P24 is

$$\left\{\tilde{v} = \left\langle\left(\frac{3725}{25}, 161, \frac{494}{3}\right); 0.6, 0.2\right\rangle, \tilde{x}_1 = \left\langle\left(\frac{19}{24}, \frac{19}{24}, \frac{19}{24}\right); 0.6, 0.2\right\rangle, \tilde{x}_2 = \left\langle\left(\frac{5}{24}, \frac{5}{24}, \frac{5}{24}\right); 0.6, 0.2\right\rangle\right\}$$

7. Concluding Remarks

We have proposed a new mathematical programming based method, called the Mehar method, for solving matrix games in which payoffs are represented by trapezoidal intuitionistic fuzzy numbers. This methods improves Li's [36] method by being based on assumptions and properties that are valid in a general case so that the new mathematical programming formulation yields a generally valid solution.

ACKNOWLEDGEMENTS

The first and second author want to acknowledge the adolescent inner blessings of Mehar (lovely daughter of the cousin sister of one of the authors, Dr. Amit Kumar). They believe that Mata Vaishno Devi has appeared on the Earth in the form of Mehar and without her blessings it was not possible to develop the ideas presented in this paper. The first author also acknowledges the financial support given to her by Department of Science and Technology under INSPIRE Programme for research students [IF130759] to complete the doctoral studies.

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