# Application of the One-factor CIR Interest Rate Model to Catastrophe Bond Pricing under Uncertainty

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**Abstract:**
*The number and amount of losses caused by natural catastrophes are important problems for insurance industry. New financial instruments were introduce to transfer risks from insurance to financial market. In this paper we consider the problem of pricing such instruments, called the catastrophe bonds (CAT bonds). We derive valuation formulas using stochastic analysis and fuzzy sets theory. As model of short interest rate we apply the one-factor Cox–Ingersoll–Ross (CIR) model. In this paper we treat the volatility of the interest rate as a fuzzy number to describe uncertainty of the market. We also apply the Monte Carlo approach to analyze the obtained cat bond fuzzy prices.*

**Keywords:** *asset pricing, catastrophe bonds, CIR model, stochastic analysis, Monte Carlo simulations, fuzzy numbers*

## 1. Introduction

Nowadays, natural catastrophes are important source of serious problems for insurers and reinsurers. Even single catastrophic event could results in damages worth of billions of dollars – e.g. the losses from Hurricane Katrina in 2005 are estimated at $40–60 billion (see [26]). The insurance industry is not prepared for such extreme damages. The classical insurance approach is based on assumption of independent and small (in comparison of the value of the whole insurance portfolio) losses (see, e.g. [3]). This assumption is not adequate in the case of outcomes of natural catastrophes, like hurricanes, floods, earthquakes etc. Therefore, after such catastrophic event, there are bankruptcies of the insurers, problems with liquidity of their reserves or increases of reinsurance premiums. For example, after Hurricane Andrew more than 60 insurance companies fell into insolvency (see [26]).

Then new kind of financial instruments were introduced. The main aim of such financial derivatives is to transfer risks from insurance markets into financial markets, which is know as *securitization* (see, e.g., [10,15,28]). Catastrophe bond, known also as cat bond or Act-of-God bond (see, e.g., [8, 14, 17, 31, 33, 38, 40]) is an example of such new approach.

The payment function of the catastrophe bond is connected with additional random variable, i.e. *triggering point*. This triggering point (indemnity trigger, parametric trigger or index trigger) depends on occurrence of specified catastrophe (like hurricane) in given region and fixed time interval or it is connected with the value of issuer's actual losses from catastrophic event (like flood), losses modeled by special software based on the real parameters of a catastrophe, or other parameters of a catastrophe or value of catastrophic losses (see, e.g. [17,40,41]). Usually if the triggering point occurs, the payments for the bondholder are lowered or even set to zero. Otherwise, the bondholder receives full payment from the cat bond.

The cat bond pricing literature is not very rich. An interesting approach applying discrete time stochastic processes within the framework of representative agent equilibrium was proposed in [9]. In [2] the authors applied compound Poisson processes to incorporate various characteristics of the catastrophe process. The authors of [5] improved and extended the method from [2]. In [1] the doubly stochastic compound Poisson process was used to model the claim index, and QMC algorithms was applied. In [13] structured cat bonds were valued with application of the indifference pricing method. Vaugirard in [40] used the arbitrage method for pricing catastrophe bonds. In his approach a catastrophe bondholder was deemed to have a short position on an option based upon a risk index. Similar approach was proposed in [25], where the Markov-modulated Poisson process was used for description of the arrival rate of natural catastrophes. In this paper we continue our earlier research concerning pricing cat bonds (see [33]). We apply stochastic analysis and fuzzy arithmetic to obtain the catastrophe bond valuation expression. In our approach the risk-free spot interest rate $r$ is described by the Cox–Ingersoll–Ross model. For description of natural catastrophe losses we use compound Poisson process with a deterministic intensity function. We also consider a complex form of catastrophe bond payoff function, which is piecewise linear. Main assumptions in our approach are: (i) the absence of arbitrage on the financial market, (ii) neutral attitude of investors to catastrophe risk. Similar assumptions were made by other authors (see, e.g. [40]).

Applying fuzzy arithmetic, we take into account different sources of uncertainty, not only the stochastic one. In particular, the volatility parameter of the spot interest rate is determined by fluctuating financial market and in many situations its uncertainty does not have stochastic type. Therefore, in order to obtain the cat bond valuation formula we apply fuzzy volatility parameter of the stochastic process $r$. As result, price obtained by us has the form of a fuzzy number. For a given $\alpha$ (e.g. $\alpha = 0.9$) its $\alpha$-level set can be used for investment decision-making as the interval of the cat bond prices with an acceptable membership degree. Similar approach was applied to option pricing in [42] and [30, 34, 35], where Jacod-Grigelionis characteristics of stochastic processes (see, e.g. [29, 39]) were additionally used. In more general setting, so called soft approaches are applied in many other fields, see, e.g. [19–23].

This paper is organized as follows. Section 2 contains preliminaries on fuzzy and interval arithmetic. In Section 3 the catastrophe bond pricing formula in crisp case for the Cox–Ingersoll–Ross risk-free interest rate model is derived. Section 4 is devoted to catastrophe bond pricing with fuzzy volatility parameter. Since the pricing formula is considered for arbitrary time moment before maturity, fuzzy random variables are additionally introduced. Apart from the fuzzy valuation formula, the expressions describing the forms of $\alpha$-level sets of the cat bond price are obtained. In Section 5 the introduced formulas are used to obtain the fuzzy prices of catastrophe bonds. Based on fuzzy arithmetic and Monte Carlo approach, the behavior of prices is analyzed for various settings close to the real-life cases. Special attention is paid to the influence of selected parameters of the model of catastrophic events on the evaluated fuzzy prices. Finally, Section 6 contains conclusions.

## 2. Fuzzy Sets Preliminaries

In this section we present basic definitions and facts concerning fuzzy and interval arithmetic, which will be used in the further part of the paper.

For a fuzzy subset $\tilde{A}$ of the set of real numbers $\mathbb{R}$ we denote by $\mu_{\tilde{A}}$ its membership function $\mu_{\tilde{A}} : \mathbb{R} \to [0,1]$ and by $\tilde{A}_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha\}$ the $\alpha$-level set of $\tilde{A}$ for $\alpha \in (0,1]$. Moreover, by $\tilde{A}_0$ we denote the closure of the set $\{x : \mu_{\tilde{A}}(x) > 0\}$.

A fuzzy number $\tilde{a}$ is a fuzzy subset of $\mathbb{R}$ for which $\mu_{\tilde{a}}$ is a normal, upper-semicontinuous, fuzzy convex function with a compact support. If $\tilde{a}$ is a fuzzy number, then for each $\alpha \in [0,1]$ the $\alpha$-level set $\tilde{a}_\alpha$ is a closed interval of the form $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$, where $\tilde{a}^L_\alpha, \tilde{a}^U_\alpha \in \mathbb{R}$ and $\tilde{a}^L_\alpha \leq \tilde{a}^U_\alpha$. We denote the set of fuzzy numbers by $\mathbb{F}(\mathbb{R})$.

Let us assume that $\odot$ is a fuzzy-number binary operator $\oplus$, $\ominus$, $\otimes$ or $\oslash$, corresponding to its real-number counterpart $\circ$: $+$, $-$, $\times$ or $/$, according to the Extension Principle.

Let $\odot_{int}$ be a binary operator $\oplus_{int}$, $\ominus_{int}$, $\otimes_{int}$ or $\oslash_{int}$ between two closed intervals $[a,b]$ and $[c,d]$. Then the following equality holds:

$$[a,b]\odot_{int}[c,d] = \{z \in R : z = x \circ y, x \in [a,b], y \in [c,d]\},$$

where $\circ$ is the corresponding real-number binary operator $+, -, \times$ or $/$, under the assumption that $0 \notin [c,d]$ in the last case. Thus, if $\tilde{a}, \tilde{b}$ are fuzzy numbers, then $\tilde{a} \odot \tilde{b}$ is also a fuzzy number and the following equalities are fulfilled.

$$(\tilde{a} \oplus \tilde{b})_\alpha = \tilde{a}_\alpha \oplus_{int} \tilde{b}_\alpha = [\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha] \,,$$

$$(\tilde{a} \ominus \tilde{b})_\alpha = \tilde{a}_\alpha \ominus_{int} \tilde{b}_\alpha = [\tilde{a}^L_\alpha - \tilde{b}^U_\alpha, \tilde{a}^U_\alpha - \tilde{b}^L_\alpha] \,,$$

$$\begin{aligned}(\tilde{a} \otimes \tilde{b})_\alpha &= \tilde{a}_\alpha \otimes_{int} \tilde{b}_\alpha = \\ &= [\min\{\tilde{a}^L_\alpha\tilde{b}^L_\alpha, \tilde{a}^L_\alpha\tilde{b}^U_\alpha, \tilde{a}^U_\alpha\tilde{b}^L_\alpha, \tilde{a}^U_\alpha\tilde{b}^U_\alpha\}, \\ &\qquad \max\{\tilde{a}^L_\alpha\tilde{b}^L_\alpha, \tilde{a}^L_\alpha\tilde{b}^U_\alpha, \tilde{a}^U_\alpha\tilde{b}^L_\alpha, \tilde{a}^U_\alpha\tilde{b}^U_\alpha\}] \,,\end{aligned}$$

$$\begin{aligned}(\tilde{a} \oslash \tilde{b})_\alpha &= \tilde{a}_\alpha \oslash_{int} \tilde{b}_\alpha = \\ &= [\min\{\tilde{a}^L_\alpha/\tilde{b}^L_\alpha, \tilde{a}^L_\alpha/\tilde{b}^U_\alpha, \tilde{a}^U_\alpha/\tilde{b}^L_\alpha, \tilde{a}^U_\alpha/\tilde{b}^U_\alpha\}, \\ &\qquad \max\{\tilde{a}^L_\alpha/\tilde{b}^L_\alpha, \tilde{a}^L_\alpha/\tilde{b}^U_\alpha, \tilde{a}^U_\alpha/\tilde{b}^L_\alpha, \tilde{a}^U_\alpha/\tilde{b}^U_\alpha\}] \,,\end{aligned}$$

if $\alpha$-level set $\tilde{b}_\alpha$ does not contain zero for all $\alpha \in [0,1]$ in the case of $\oslash$.

A fuzzy number $\tilde{a}$ is called positive ($\tilde{a} \geq 0$) if $\mu_{\tilde{a}}(x) = 0$ for $x < 0$ and it is called strictly positive ($\tilde{a} > 0$) if $\mu_{\tilde{a}}(x) = 0$ for $x \leq 0$.

A triangular fuzzy number $\tilde{a} = (a_1, a_2, a_3)$ is a fuzzy number with the membership function of the form

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\ \frac{x-a_3}{a_2-a_3} & \text{if } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise.} \end{cases}$$

In our further considerations we will use the following proposition, proved in [42].

**Proposition 1.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f^{-1}(\{y\})$ is a compact set for each $y \in \mathbb{R}$. Then $f$ induces a fuzzy-valued function $\tilde{f} : \mathbb{F}(\mathbb{R}) \to \mathbb{F}(\mathbb{R})$ via the Extension Principle and for each $\tilde{\Lambda} \in \mathbb{F}(\mathbb{R})$ the $\alpha$-level set of $\tilde{f}(\tilde{\Lambda})$ has the form $\tilde{f}(\tilde{\Lambda})_\alpha = \{f(x) : x \in \tilde{\Lambda}_\alpha\}$.*

We recall the notions of weighted interval-valued and crisp possibilistic mean values of fuzzy numbers. For details we refer the reader to [16].

Let $\tilde{a} \in \mathbb{F}(\mathbb{R})$. A non-negative, monotone increasing function $f : [0,1] \mapsto \mathbb{R}$ such that $\int_0^1 f(\alpha)d\alpha = 1$ is said to be a weighting function. The lower and upper weighted possibilistic mean values $M_*(\tilde{a})$ and $M^*(\tilde{a})$ of $\tilde{a}$ are defined by the integrals:

$$M_*(\tilde{a}) = \int_0^1 \tilde{a}^L_\alpha f(\alpha)d\alpha,$$

$$M^*(\tilde{a}) = \int_0^1 \tilde{a}^U_\alpha f(\alpha)d\alpha.$$

The weighted interval-valued possibilistic mean $M(\tilde{a})$ and the crisp weighted possibilistic mean $\bar{M}(\tilde{a})$ of the fuzzy number $\tilde{a}$ have the following form:

$$M(\tilde{a}) = [M_*(\tilde{a}), M^*(\tilde{a})],$$

$$\bar{M}(\tilde{a}) = \frac{M_*(\tilde{a}) + M^*(\tilde{a})}{2}.$$

Let $\mathcal{B}(\mathbb{R})$ be the Borel $\sigma$-field of subsets of $\mathbb{R}$ and $(\Omega, \mathcal{F})$ be a measurable space. A fuzzy-number-valued map $\tilde{X} : \Omega \mapsto \mathbb{F}(\mathbb{R})$ is called a fuzzy random variable if

$$\left\{(\omega, x) : \tilde{X}(\omega)(x) \geq \alpha\right\} \in \mathcal{F} \times \mathcal{B}(\mathbb{R})$$

for every $\alpha \in [0,1]$ (see, e.g. [36]).

## 3. Catastrophe Bond Pricing in Crisp Case

As it was previously noted, the triggering point changes the structure of the payment function of the cat bond. Usually cat bonds are issued by insurers or reinsurers (see, e.g., [37]) via a special tailor-made fund, called a special purpose vehicle (SPV) or special purpose company (SPC) (see, e.g., [24, 40]). The hedger (e.g. insurer or reinsurer) pays an insurance premium in exchange for coverage in the case if triggering point occurs (see Figure 1). The investors purchase the catastrophe bonds for cash. The premium and cash flows are directed to SPV, which purchases safe securities and issues the catastrophe bonds. Investors hold these assets whose payments depend on occurrence of the triggering point. If the pre-specified event occurs during the fixed period (e.g. there is a specified kind of natural catastrophe), the SPV compensates the insurer and the cash flows for investors are changed. Usually these flows are lowered, i.e. there is full or partial forgiveness of the payment. However, if the triggering point does not occur, the investors usually receive the full payment (i.e. the face value of the bond).

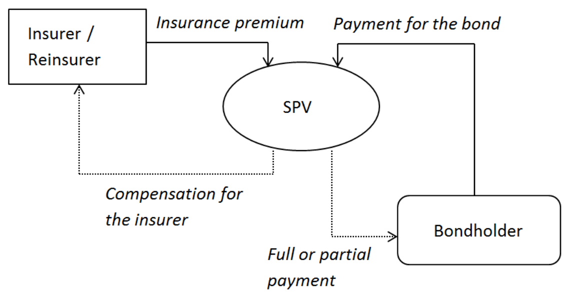


***Fig. 1. Payments related to issuing and terminating of the cat bond***

In the further part of this section we derive and present the pricing formula for catastrophe bonds in crisp case, assuming no arbitrage opportunity on the market. At the beginning we introduce all necessary definitions and assumptions.

We use stochastic processes with continuous time to describe the dynamics of the spot risk-free interest rate and the cumulative catastrophe losses. The time horizon has the form $[0, T']$, where $T' > 0$. The date of maturity of catastrophe bonds $T$ is not later than $T'$, i.e. $T \leq T'$. We consider two probability measures: $P$ and $Q$ and denote the expected values with respect to them by the symbols $E^P$ and $E^Q$.

We introduce standard Brownian motion $(W_t)_{t\in[0,T']}$ and Poisson process $(N_t)_{t\in[0,T']}$ with a deterministic intensity function $\rho(t), t \in [0, T']$. The Brownian motion will be used for description of the risk-free interest rate.

We introduce a sequence $(U_i)_{i=1}^{\infty}$ of independent, identically distributed random variables with finite second moment. For each $i$ the random variable $U_i$ will describe the value of losses during $i$-th catastrophic event.

We define compound Poisson process by the formula

$$\tilde{N}_t = \sum_{i=1}^{N_t} U_i, t \in [0, T'] .$$

for modeling the cumulative catastrophic losses till moment $t$.

All the introduced above processes and random variables are defined on probability space $(\Omega, \mathcal{F}, P)$. We introduce the following filtrations: $(\mathcal{F}_t^0)_{t\in[0,T']}$, $(\mathcal{F}_t^1)_{t\in[0,T']}$ and $(\mathcal{F}_t)_{t\in[0,T']}$. $(\mathcal{F}_t^0)_{t\in[0,T']}$ is generated by $W$, $(\mathcal{F}_t^1)_{t\in[0,T']}$ by $\tilde{N}$ and $(\mathcal{F}_t)_{t\in[0,T']}$ by $W$ and $\tilde{N}$. Moreover, they are augmented to encompass $P$-null sets from $\mathcal{F}_{T'}^0$, $\mathcal{F}_{T'}^1$ and $\mathcal{F} = \mathcal{F}_{T'}$, respectively.

$(W_t)_{t\in[0,T']}$, $(N_t)_{t\in[0,T']}$ and $(U_i)_{i=1}^{\infty}$ are independent and the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T']}, P\right)$ satisfies usual assumptions: $\sigma$-algebra $\mathcal{F}$ is $P$-complete, the filtration $(\mathcal{F}_t)_{t\in[0,T']}$ is right continuous and each $\mathcal{F}_t$ contains all the $P$-null sets from $\mathcal{F}$.

Let $r = (r_t)_{t\in[0,T']}$ be the risk-free spot interest rate, i.e. short-term rate for risk-free borrowing or lending at time $t$ over the infinitesimal time interval $[t, t + dt]$. We assume that $r$ is an one-factor affine model. For more details concerning affine interest rate models we refer the reader to [11] and [12]. The Cox – Ingersoll – Ross model, considered in this paper, is of this type.

The risk-free spot interest rate $(r_t)_{t\in[0,T']}$, belonging to the class of one-factor affine models, is a diffusion process of the form

$$dr_t = \boldsymbol{\alpha}(r_t)\, dt + \boldsymbol{\sigma}(r_t)\, dW_t, \quad (1)$$

where

$$\boldsymbol{\alpha}(r) = \varphi - \kappa r \text{ and } \boldsymbol{\sigma}^2(r) = \delta_1 + \delta_2 r$$

for constants $\varphi, \kappa, \delta_1, \delta_2$ (see, e.g. [27]). We denote by $\mathcal{S}$ the set of all the values which $r$ can have with strictly positive probability. We require that $\delta_1 + \delta_2 r \geq 0$ for all values $r \in \mathcal{S}$.

We assume that zero-coupon bonds are traded on the market, investors have neutral attitude to catastrophe risk and interest rate changes are replicable by other financial instruments. Moreover, we assume that there is no arbitrage opportunity on the market. Then the family of zero-coupon bonds prices is arbitrage-free with respect to $r$ for the probability measure $Q$ equivalent to $P$, given by the Radon-Nikodym derivative

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \bar{\lambda}_t dW_t - \frac{1}{2}\int_0^T \bar{\lambda}_t^2 dt\right), P - a.s. \quad (2)$$

where $\bar{\lambda}_t = \lambda_0 \boldsymbol{\sigma}(r_t)$ is the market price of risk process, $\lambda_0 \in \mathbb{R}$. Under $Q$ the process $r$ is described by

$$dr_t = \hat{\boldsymbol{\alpha}}(r_t)\, dt + \boldsymbol{\sigma}(r_t)\, dW_t^Q, \quad (3)$$

where

$$\hat{\boldsymbol{\alpha}}(r) = \hat{\varphi} - \hat{\kappa} r,\ \hat{\varphi} = \varphi - \lambda_0 \delta_1, \hat{\kappa} = \kappa + \lambda_0 \delta_2 \quad (4)$$

and $W_t^Q$ is $Q$ - Brownian motion.

We fix $n \geq 1$, $T \in [0, T']$ and $Fv > 0$. Let $K = (K_0, K_1, ..., K_n)$ be levels of catastrophic losses, where

$$0 \leq K_0 < K_1 < K_1 < ... < K_n.$$

Let $w = (w_1, w_2, ..., w_n)$ be a sequence of non-negative numbers such that their sum is not greater to 1, i.e. $\sum_{i=1}^{n} w_i \leq 1$.

**Definition 1.** *By the symbol $IB(T, Fv)$ we denote catastrophe bond with the face value $Fv$, the date of maturity and payoff $T$ and the payoff function of the form*

$$\nu_{T,Fv} = Fv\left(1 - \sum_{j=0}^{n-1} \frac{\tilde{N}_T \wedge K_{j+1} - \tilde{N}_T \wedge K_j}{K_{j+1} - K_j} w_{j+1}\right).$$

For the considered type of catastrophe bond the payoff function is a piecewise linear function of $\tilde{N}_T$. If the catastrophe does not occur (i.e. $\tilde{N}_T < K_0$), the bondholder receives the payoff equal to its face value $Fv$. If $\tilde{N}_T \geq K_n$, the payoff is equal to $Fv\left(1 - \sum_{i=1}^{n} w_i\right)$. If $K_j \leq \tilde{N}_T \leq K_{j+1}$ for $j = 0, 1, ..., n$, the bondholder is paid

$$Fv\left(1 - \sum_{0 \leq i < j} w_{i+1} - \frac{\tilde{N}_T \wedge K_{j+1} - \tilde{N}_T \wedge K_j}{K_{j+1} - K_j} w_{j+1}\right)$$

and in the interval $[K_j, K_{j+1}]$ the payoff decreases linearly from $Fv\left(1 - \sum_{0 \leq i < j} w_{i+1}\right)$ to $Fv\left(1 - \sum_{0 \leq i \leq j} w_{i+1}\right)$ as the function of $\tilde{N}_T$.

We will use the following general theorem concerning catastrophe bond pricing, proved by us in [32].

**Theorem 1.** *Let $(r_t)_{t \in [0, T']}$ be a risk-free spot interest rate given by the diffusion process (1) and such that, after the change of probability measure described by the Radon – Nikodym derivative (2), it has the form (3) with the coefficients given by equalities (4). Let $IB_{T,Fv}(t)$ be the price at time t, $0 \leq t \leq T$, of the catastrophe bond $IB(T, Fv)$. Then*

$$IB_{T,Fv}(t) = \eta(t, T, r_t, Fv), 0 \leq t \leq T, \quad (5)$$

*where*

**(i)**

$$\eta(t, T, r, Fv) = \\ = \exp(-a(T-t) - b(T-t)r) E^Q\left(\nu_{T,Fv} | \mathcal{F}_t^1\right); \quad (6)$$

**(ii)** *functions $a(\tau)$ and $b(\tau)$ satisfy the following system of differential equations:*

$$\frac{1}{2}\delta_2 b^2(\tau) + \hat{\kappa} b(\tau) + b'(\tau) - 1 = 0, \quad \tau > 0, \quad (7)$$

$$a'(\tau) - \hat{\varphi} b(\tau) + \frac{1}{2}\delta_1 b^2(\tau) = 0 \quad \tau > 0$$

*with $a(0) = b(0) = 0$.*

*In particular,*

$$IB_{T,Fv}(0) = \eta(0, T, r_0, Fv) \\ = \exp(-a(T) - b(T) r_0) E^P \nu_{T,Fv}. \quad (8)$$

The interest rate process $r$, applied in this paper, is the Cox–Ingersoll–Ross model described by the following stochastic equation

$$dr_t = \kappa(\theta - r_t) dt + \Gamma\sqrt{r_t} dW_t \quad (9)$$

for positive constants $\kappa, \theta$ and $\Gamma$. The CIR model is affine with parameters $\varphi = \kappa\theta$, $\delta_1 = 0$ and $\delta_2 = \Gamma^2$. Generally, for the considered model, interest rate cannot become negative (i.e., $\mathcal{S} = [0, \infty)$), which is a major advantage relative to other models. Moreover, if its parameters satisfy the inequality $2\varphi \geq \Gamma^2$, then $\mathcal{S} = (0, \infty)$. The CIR model has the property of mean reversion around the long-term level $\theta$. The parameter $\kappa$ controls the size of the expected adjustment towards $\theta$ and is called the speed of adjustment. The volatility is the product $\Gamma\sqrt{r_t}$ and therefore the interest rate is less volatile for low values than for high values of the process $r_t$.

The following theorem is a special case of Theorem 1 for the spot interest rate dynamics described by the Cox–Ingersoll–Ross model.

**Theorem 2.** *Let the risk-free spot interest rate $(r_t)_{t \in [0, T']}$ be described by the CIR model. Assume that $IB_{T,Fv}(t)$ is the price of the bond $IB(T, Fv)$ at moment $t \in [0, T]$. Then*

$$IB_{T,Fv}(t) = e^{a(T-t) - b(T-t) r_t} E^Q\left(\nu_{T,Fv} | \mathcal{F}_t^1\right), \quad (10)$$

*where*

$$b(\tau) = \frac{(e^{\gamma\tau} - 1)}{\frac{(\hat{\kappa}+\gamma)}{2}(e^{\gamma\tau} - 1) + \gamma}, \quad (11)$$

$$a(\tau) = \frac{2\varphi}{\Gamma^2}\left[\ln\left(\frac{\gamma}{\frac{(\hat{\kappa}+\gamma)}{2}(e^{\gamma\tau} - 1) + \gamma}\right) + \frac{(\hat{\kappa}+\gamma)\tau}{2}\right], \quad (12)$$

$\hat{\kappa} = \kappa + \lambda\Gamma$, $\gamma = \sqrt{\hat{\kappa}^2 + 2\Gamma^2}$.

In Theorem 2 the constant $\lambda$ is the product $\lambda = \lambda_0\Gamma$. Since all the model parameters should be positive after change of probability measure, we assume that $\hat{\kappa} > 0$. The equalities (11) and (12) are obtained as the solution of the system of equations (7). One can also find them in financial literature (see, e.g. [27]), since they are used in the zero-coupon bond pricing formula.

## 4. Catastrophe Bond Pricing in Fuzzy Case

Usually some parameters of the financial market are not precisely known. In particular, the volatility parameter of the spot interest rate is determined by fluctuating financial market and very often its uncertainty does not have stochastic character. Therefore it is unreasonable to choose fixed values of parameters, which are obtained from historical data, for later use in pricing model, since they can fluctuate in future (see, e.g. [42]).

To estimate values of uncertain parameters one can use knowledge of experts, asking them for forecast of a parameter. The forecasts can be transferred into triangular fuzzy numbers. Their average can be computed and used for estimation of the parameter. Such an estimation method was proposed in [4] and [18] for financial applications.

In the reminder of this paper we assume more generally that the volatility parameter is a strictly positive fuzzy number, which is not necessarily triangular. We denote the fuzzy volatility parameter by $\tilde{\Gamma}$.

In the following theorem we present catastrophe bonds pricing formula for the one-factor Cox–Ingersoll–Ross interest rate model.

**Theorem 3.** *Assume that $IB_{T,Fv}(t)$ is the price of bond $IB(T,Fv)$ at moment $t\in[0,T]$ for a strictly positive fuzzy volatility parameter $\tilde{\Gamma}$. Then*

$$IB_{T,Fv}(t)=e^{\tilde{a}(T-t)\ominus\tilde{b}(T-t)\otimes\tilde{r}_t}\otimes E^Q\left(\nu_{T,Fv}|\mathcal{F}_t^1\right), \tag{13}$$

*where*

$$\tilde{a}(\tau)=\tilde{\phi}\otimes\tilde{\delta}(\tau),\ \tilde{b}(\tau)=\tilde{\alpha}(\tau)\oslash\tilde{\beta}(\tau),$$

$$\tilde{\phi}=(2\varphi)\oslash\left(\tilde{\Gamma}\otimes\tilde{\Gamma}\right),\ \tilde{\kappa}=\kappa\oplus\lambda\otimes\tilde{\Gamma}>0,$$

$$\tilde{\gamma}=\sqrt{\tilde{\kappa}\otimes\tilde{\kappa}\oplus 2\otimes\tilde{\Gamma}\otimes\tilde{\Gamma}},$$

$$\tilde{\alpha}(\tau)=e^{\tilde{\gamma}\otimes\tau}\ominus 1,\ \tilde{\beta}(\tau)=\frac{1}{2}\otimes\tilde{\alpha}(\tau)\otimes(\tilde{\kappa}\oplus\tilde{\gamma})\oplus\tilde{\gamma}$$

*and*

$$\tilde{\delta}(\tau)=\ln\left(\tilde{\gamma}\oslash\tilde{\beta}(\tau)\right)\oplus\frac{\tau}{2}\otimes(\tilde{\kappa}\oplus\tilde{\gamma}).$$

*Moreover, for $\alpha\in[0,1]$,*

$$\left(IB_{T,Fv}(t)\right)_\alpha= \tag{14}$$
$$\left[E^Q\left(\nu_{T,Fv}|\mathcal{F}_t^1\right)e^{(\tilde{a}(T-t))_\alpha^L-\left(\tilde{b}(T-t)\right)_\alpha^U(r_t)_\alpha^U},\right.$$
$$\left.E^Q\left(\nu_{T,Fv}|\mathcal{F}_t^1\right)e^{(\tilde{a}(T-t))_\alpha^U-\left(\tilde{b}(T-t)\right)_\alpha^L(r_t)_\alpha^L}\right],$$

*where*

$$\tilde{\kappa}_\alpha=\begin{cases}\left[\lambda\tilde{\Gamma}_\alpha^L+\kappa,\lambda\tilde{\Gamma}_\alpha^U+\kappa\right] & \textit{for } \lambda>0,\\ \left[\lambda\tilde{\Gamma}_\alpha^U+\kappa,\lambda\tilde{\Gamma}_\alpha^L+\kappa\right] & \textit{for } \lambda<0,\\ \kappa & \textit{for } \lambda=0,\end{cases} \tag{15}$$

$$(\tilde{\kappa}\otimes\tilde{\kappa})_\alpha=\begin{cases}\left[\left(\lambda\tilde{\Gamma}_\alpha^L+\kappa\right)^2,\left(\lambda\tilde{\Gamma}_\alpha^U+\kappa\right)^2\right] & \textit{for } \lambda>0,\\ \left[\left(\lambda\tilde{\Gamma}_\alpha^U+\kappa\right)^2,\left(\lambda\tilde{\Gamma}_\alpha^L+\kappa\right)^2\right] & \textit{for } \lambda<0,\\ \kappa^2 & \textit{for } \lambda=0,\end{cases} \tag{16}$$

$$\tilde{\gamma}_\alpha=\left[\sqrt{(\tilde{\kappa}\otimes\tilde{\kappa})_\alpha^L+2\left(\tilde{\Gamma}_\alpha^L\right)^2},\sqrt{(\tilde{\kappa}\otimes\tilde{\kappa})_\alpha^U+2\left(\tilde{\Gamma}_\alpha^U\right)^2}\right], \tag{17}$$

$$(\tilde{\alpha}(\tau))_\alpha=\left[e^{\tilde{\gamma}_\alpha^L\tau}-1,e^{\tilde{\gamma}_\alpha^U\tau}-1\right],$$

$$\tilde{\phi}_\alpha=\left[\frac{2\varphi}{\left(\tilde{\Gamma}_\alpha^U\right)^2},\frac{2\varphi}{\left(\tilde{\Gamma}_\alpha^L\right)^2}\right], \tag{18}$$

$$\left(\tilde{\delta}(\tau)\right)_\alpha=\left[\ln\left(\frac{\tilde{\gamma}_\alpha^L}{\frac{1}{2}(\tilde{\alpha}(\tau))_\alpha^U(\tilde{\kappa}_\alpha^U+\tilde{\gamma}_\alpha^U)+\tilde{\gamma}_\alpha^U}\right)\right.$$
$$+\frac{\tau\left(\tilde{\kappa}_\alpha^L+\tilde{\gamma}_\alpha^L\right)}{2},\ln\left(\frac{\tilde{\gamma}_\alpha^U}{\frac{1}{2}(\tilde{\alpha}(\tau))_\alpha^L(\tilde{\kappa}_\alpha^L+\tilde{\gamma}_\alpha^L)+\tilde{\gamma}_\alpha^L}\right)$$
$$\left.+\frac{\tau\left(\tilde{\kappa}_\alpha^U+\tilde{\gamma}_\alpha^U\right)}{2}\right], \tag{19}$$

$$\left(\tilde{b}(\tau)\right)_\alpha=\left[\frac{(\tilde{\alpha}(\tau))_\alpha^L}{\frac{1}{2}(\tilde{\alpha}(\tau))_\alpha^U(\tilde{\kappa}_\alpha^U+\tilde{\gamma}_\alpha^U)+\tilde{\gamma}_\alpha^U},\right.$$
$$\left.\frac{(\tilde{\alpha}(\tau))_\alpha^U}{\frac{1}{2}(\tilde{\alpha}(\tau))_\alpha^L(\tilde{\kappa}_\alpha^L+\tilde{\gamma}_\alpha^L)+\tilde{\gamma}_\alpha^L}\right] \tag{20}$$

*and*

$$(a(\tau))_\alpha=\left[\left(\tilde{\phi}_\alpha\otimes_{int}\left(\tilde{\delta}(\tau)\right)_\alpha\right)^L,\right.$$
$$\left.\left(\tilde{\phi}_\alpha\otimes_{int}\left(\tilde{\delta}(\tau)\right)_\alpha\right)^U\right]. \tag{21}$$

*Proof.* We replace the crisp volatility parameter $\Gamma$ by its fuzzy counterpart $\tilde{\Gamma}$ and arithmetic operators $+,-,\cdot$ by $\oplus,\ominus,\otimes$ in (10). As result we obtain the formula (13).

Let $\alpha\in[0,1]$ and $\tau\geq 0$. For a given fuzzy number $\tilde{F}$ we denote by $\tilde{F}_\alpha^L$ and $\tilde{F}_\alpha^U$ the lower and upper bound of its $\alpha$-level set.

Since $\varphi,\kappa>0$ and $\tilde{\Gamma}>0$, the number $\tilde{\kappa}\otimes\tilde{\kappa}\oplus 2\otimes\tilde{\Gamma}\otimes\tilde{\Gamma}$ is also strictly positive. From direct calculations it follows that (15) and (16) hold.

Function $\exp(x)$ for $x\in\mathbb{R}$ and functions $\sqrt{x}$ and $\ln(x)$ for $x>0$ satisfy the assumptions of Proposition 1 and they are increasing.
Thus, $\tilde{\gamma}>0$,

$$\tilde{\gamma}_\alpha=\left[\sqrt{\left(\tilde{\kappa}\otimes\tilde{\kappa}\oplus 2\otimes\tilde{\Gamma}\otimes\tilde{\Gamma}\right)_\alpha^L},\sqrt{\left(\tilde{\kappa}\otimes\tilde{\kappa}\oplus 2\otimes\tilde{\Gamma}\otimes\tilde{\Gamma}\right)_\alpha^U}\right]$$

and (17) is satisfied. From direct calculations it follows that $\tilde{\kappa}\oplus\tilde{\gamma}>0$, $\tilde{\alpha}(\tau)\geq 0$, $\tilde{\beta}(\tau)>0$, $\tilde{b}(\tau)\geq 0$ and (20) is fulfilled. Proposition 1 implies the equality

$$\left(e^{\tilde{a}(\tau)\ominus\tilde{b}(\tau)\otimes r_t}\right)_\alpha=\left[e^{\left(\tilde{a}(\tau)\ominus\tilde{b}(\tau)\otimes\tilde{r}_t\right)_\alpha^L},\right.$$
$$\left.e^{\left(\tilde{a}(\tau)\ominus\tilde{b}(\tau)\otimes\tilde{r}_t\right)_\alpha^U}\right], \tag{22}$$

>From properties of the Cox–Ingersoll–Ross interest rate model it follows that the fuzzy random variable $\tilde{r}_t$ is positive for $t\in[0,T]$ and, since $\tilde{b}(T-t)\geq 0$,

that (14) holds. Applying Proposition 1 also gives the equality

$$\left(\ln\left(\tilde{\gamma}\oslash\tilde{\beta}(\tau)\right)\right)_\alpha$$
$$=\left[\ln\left(\frac{\tilde{\gamma}_\alpha^L}{\frac{1}{2}\left(\tilde{\alpha}(\tau)\right)_\alpha^U\left(\tilde{\kappa}_\alpha^U+\tilde{\gamma}_\alpha^U\right)+\tilde{\gamma}_\alpha^U}\right),\right.$$
$$\left.\ln\left(\frac{\tilde{\gamma}_\alpha^U}{\frac{1}{2}\left(\tilde{\alpha}(\tau)\right)_\alpha^L\left(\tilde{\kappa}_\alpha^L+\tilde{\gamma}_\alpha^L\right)+\tilde{\gamma}_\alpha^L}\right)\right],$$

Finally, the standard interval calculations give the forms of $\tilde{\phi}_\alpha$, $\left(\tilde{\delta}(\tau)\right)_\alpha$ and $(a(\tau))_\alpha=\left(\tilde{\phi}\otimes\tilde{\delta}(\tau)\right)_\alpha$ described by (18), (19) and (21). □

Applying the equality

$$\mu_{I\tilde{B}_{T,Fv}(t)}(c)=\sup_{0\le\alpha\le1}\alpha I_{\left(I\tilde{B}_{T,Fv}(t)\right)_\alpha}(c)$$

one can obtain the membership function of $I\tilde{B}_{T,Fv}(t)$. For a sufficiently high value of $\alpha$ (e.g. $\alpha=0.95$) the $\alpha$-level set of $I\tilde{B}_{T,Fv}(t)$ can be used for investment decision-making. A financial analyst can choose any value from the $\alpha$-level set as the catastrophe bond price with an acceptable membership degree.

## 5. Monte Carlo Approach

The calculations required to find the price of the cat bond via the formulas introduced in Section 4 could be very complex, especially if the payment function or the model of losses are not straightforward ones. Then instead of directly finding an analytical formula for the price, other approaches may be used. In this paper we focus on Monte Carlo simulations and application of fuzzy arithmetic for $\alpha$-cuts.

To model complex nature of the practical cases, the parameters similar to the ones based on the real-life data are applied. In [6] the parameters of the CIR model are estimated using Kalman filter for monthly data of the Treasury bond market. But these values, namely $\varphi,\kappa,\Gamma,r_0$, are crisp ones (compare with Table 1). Because in Section 4 the cat bond pricing approach for the CIR model with fuzzy number $\tilde{\Gamma}$ was established, then instead of crisp value $\Gamma=0.0754$ (as estimated in [6]), the fuzzy triangular number $\tilde{\Gamma}$ is applied (see Table 1). This fuzzy value is similar to the crisp parameter obtained in [6], but with the introduced fuzzy volatility the future uncertainty of the financial markets is modeled.

The other applied model, i.e. the process of losses, is also based in our approach on the real-life data. In [7] the information of catastrophe losses in the United States provided by the Property Claim Services (PCS) of the ISO (Insurance Service Office Inc.) and the relevant estimation procedure for this data are considered. For each catastrophe, the PCS loss estimate represents anticipated industrywide insurance payments for different property lines of insurance covering. An event is noted as a catastrophe when claims are expected to reach a certain dollar threshold. We focus on lognormal distribution of the value of the single loss and NHPP (non-homogeneous Poisson process) as the process of the quantity of catastrophic events (see Table 1), but other random distributions and other processes could be directly applied using the approach introduced in this paper.

As noted in [7], because of annual seasonality of occurrence of catastrophic events, the intensity function of losses for NHPP is given by

$$\rho_{\mathrm{NHPP}}(t)=a+2\pi b\sin\left(2\pi(t-c)\right). \quad (23)$$

The triggering points in our considerations are related to quantiles given by $Q_{\text{NHPP-LN}}(x)$, i.e. the $x$-th quantile of the cumulated value of losses for the NHPP process (quantity of losses) and lognormal distribution (value of each loss).

After conducting $N=1000000$ Monte Carlo simulations, the fuzzy value of the cat bond price was obtained using fuzzy arithmetic (see Figure 2). This fuzzy price is close to symmetry in the case of the parameters from Table 1. Based on this fuzzy number, the relevant intervals of prices for various $\alpha$ may be also found. Because of practical purposes the analyst may be also interested in crisp value of the cat bond price, then e.g. $\alpha=1$ can be set or the crisp possibilistic mean can be calculated (see [34, 35] for related approach in analysis of European options pricing). The obtained results in the considered case are enumerated in Table 2. For the crisp possibilistic mean the intuitive function $f(\alpha)=2\alpha$ is applied. The difference between both of the obtained crisp values is about 0.091%.

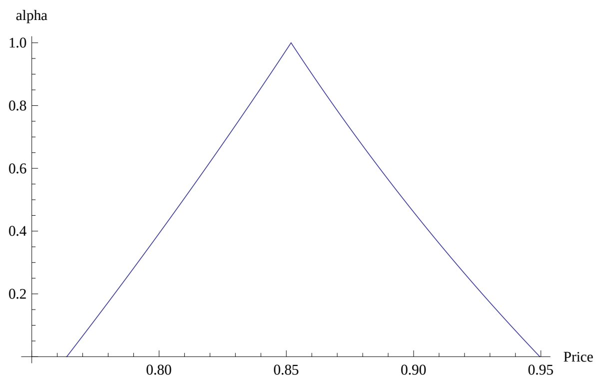


***Fig. 2. Fuzzy price of the cat bond (parameters of the model from Table 1)***

For other symmetric fuzzy values of the volatility $\tilde{\Gamma}$ considered in our analysis, the calculated fuzzy cat bond prices have similar shapes (see Figure 3). The membership function could be also evaluated in the case of asymmetrical triangular fuzzy values of $\tilde{\Gamma}$ (see Figure 4).

The model of catastrophic events is usually based on historical data as in the case discussed in [7]. Therefore the estimators calculated from such data may be not completely adequate for future natural catastrophes. Then the behavior of cat bond prices could be analyzed if some of the important parameters of the model are changed. For example, if the parameter $\mu_{\mathrm{LN}}$ of lognormal distribution of the single loss becomes

***Tab. 1. Parameters of Monte Carlo simulations***

| | Parameters |
|---|---|
| CIR model (crisp) | $\varphi = 0.00270068, \kappa = 0.07223, r_0 = 0.02$ |
| CIR model (fuzzy) | $\tilde{\Gamma} = (0.07, 0.075, 0.08)$ |
| Intensity of NHPP | $a = 30.875, b = 1.684, c = 0.3396$ |
| Lognormal distribution | $\mu_{\text{LN}} = 17.357, \sigma_{\text{LN}} = 1.7643$ |
| Triggering points | $K_1 = Q_{\text{NHPP-LN}}(0.75), K_2 = Q_{\text{NHPP-LN}}(0.85), K_3 = Q_{\text{NHPP-LN}}(0.95)$ |
| Values of losses coefficients | $w_1 = 0.4, w_2 = 0.6$ |

***Tab. 2. Crisp prices for the cat bond (parameters of the model from Table 1)***

| Method | Price |
|---|---|
| $\alpha = 1$ | 0.851857 |
| Crisp possibilistic mean | 0.852631 |

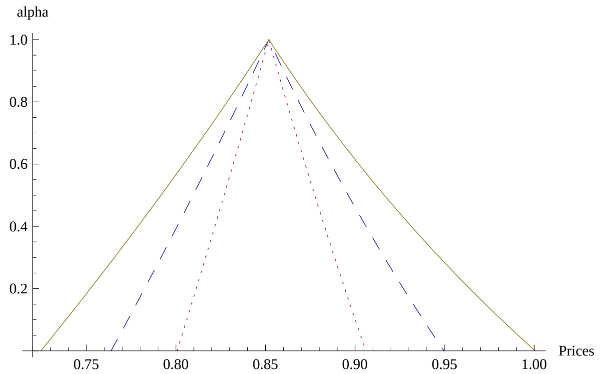


***Fig. 3. Fuzzy price of the cat bond for various fuzzy values of $\tilde{\Gamma}$ ($(0.072, 0.075, 0.078)$ – dotted line, $(0.07, 0.075, 0.08)$ – dashed line, $(0.068, 0.075, 0.082)$ – solid line)***

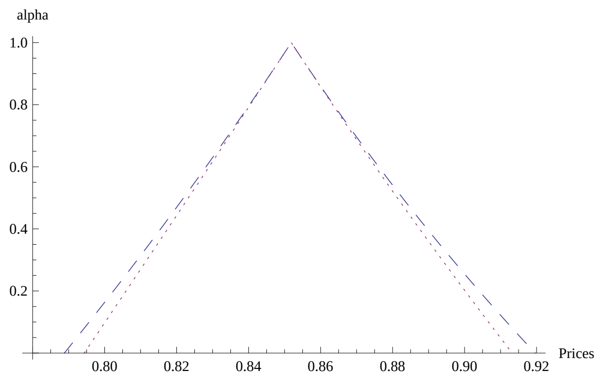


***Fig. 4. Fuzzy price of the cat bond for various fuzzy values of $\tilde{\Gamma}$ ($(0.07, 0.075, 0.077)$ – dotted line, $(0.073, 0.075, 0.08)$ – dashed line)***

higher and other parameters are the same as in Table 1, then the relevant fuzzy prices of the cat bonds are shifted left-side (see Figure 5) and the crisp prices are lower (see Table 3). The same applies for the case of various values of the parameter $\sigma_{\text{LN}}$ (see Figure 6 and Table 4). As it may be seen from Figure 5 and Figure 6, these parameters have important impact on the obtained cat bond prices.

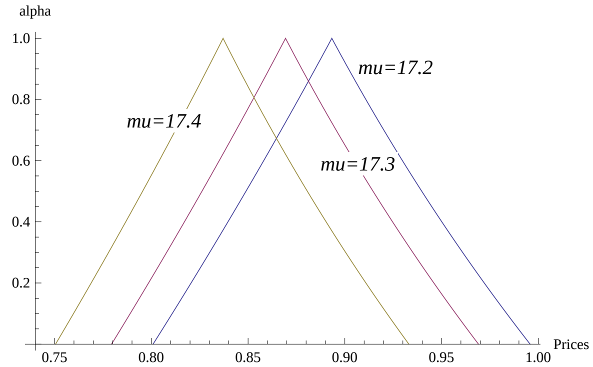


***Fig. 5. Fuzzy price of the cat bond for various values of $\mu_{LN}$***

***Tab. 3. Crisp prices for for various values of $\mu_{LN}$***

| $\mu_{\text{LN}}$ | 17.2 | 17.3 | 17.4 |
|---|---|---|---|
| Price for $\alpha = 1$ | 0.893286 | 0.86935 | 0.83707 |
| Crisp possibilistic mean | 0.894098 | 0.87014 | 0.837831 |

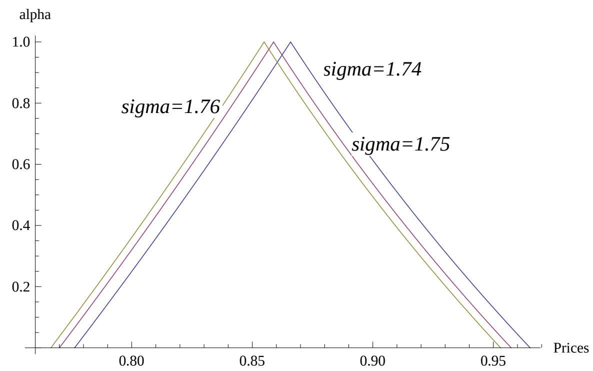


***Fig. 6. Fuzzy price of the cat bond for various values of $\sigma_{LN}$***

## 6. Conclusions

In this paper the catastrophe bond pricing formula in crisp case for the Cox–Ingersoll–Ross risk-free interest rate model is derived. Then on basis of this formula catastrophe bond valuation expression for fuzzy

*Tab. 4. Crisp prices for for various values of $\sigma_{LN}$*

| $\sigma_{\mathrm{LN}}$ | 1.74 | 1.75 | 1.76 |
|---|---|---|---|
| Price for $\alpha = 1$ | 0.865891 | 0.858843 | 0.854912 |
| Crisp possibilistic mean | 0.866678 | 0.859623 | 0.855689 |

volatility parameter is obtained. Since the pricing formula is considered for arbitrary time moment before maturity, fuzzy random variables are introduced. Besides the fuzzy valuation formula, the forms of $\alpha$-level sets of the cat bond price are received. Therefore this approach can be applied for general forms of fuzzy numbers.

Also the Monte Carlo simulations are conducted in order to directly analyze the fuzzy cat bond prices. We apply fuzzy arithmetic and introduce triangular fuzzy number for the value of the volatility in CIR model, but using other fuzzy numbers (e.g. L-R numbers) is also possible in our setting. Then the influence of the shape of fuzzy numbers and other parameters of the model like distribution of the single loss on the final cat bond price is considered.

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