# Nonparametric Identification Method of Stochastic Differential Equation with Fractal Brownian Motion

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**Abstract:**

*This Paper presents a methodology for estimating the parameters of stochastic differential equation (SDE) driven by fractional Brownian motion (fBm). The main idea is connected with simulated maximum likelihood. To develop the methodology, two important questions, namely how to generate fBm sample paths with different values of the Hurst parameter and how to estimate Hurst parameter are studied. An Effectiveness of the methodology is analyzed through Monte Carlo simulations.*


***Keywords:*** *parameters estimation, identification*

## 1. Introduction

Any natural phenomena can be modeled by SDEs driven by stochastic processes of different nature. Indeed, SDEs find application in many disciplines including telecommunication, engineering, economics and finance, biology, physics and medicine. Although the analysis of SDE has received attention over a long period, the estimation of their parameters when the process differs from white noise and is observed at discrete instants only, until recently, received less attention [1] – [4].

### *1.1. Some properties of fractional brownian motion*

The fBm of Hurst parameter $H \in\ ]0,1[$ is a Gaussian, mean-zero and $H$ -self-similar process with $B_0^H = 0$ and stationary increments. It can be defined as $B^H = \{B_t^H, t \in [t_0, t_T]\}$ with covariance function of the form

$$Cov\left(B_s^H, B_t^H\right) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), 0 \le s < t \quad (1)$$

The trajectories of  are almost surely continuous and not differentiable.

Moreover for any $t \ge 0$ $E\left[\left(B_t^H\right)^2\right] = |t|^{2H}$. If $H = 1/2$, $B^H$ is the usual white noise denoted by $B = \{B_t, t \in [t_0, t_T]\}$.

The fBm is widely used in different fields of science as far as there is a possibility to present long-range and short range dependent processes. However, there are some difficulties connected with the fact that fBm is not a semi martingale and the paths of fBm are of unbounded variation. Thus the usual Lebesgue-Stieltjes integration and Ito stochastic calculus cannot be applied. This complicates the development of new pathwise integration method with respect to fBm, the numerical schemes for SDE driven by fBm and especially if one is interested in simulation, where mathematical model contains fBm [5].

### *1.2. Semiparametric estimator of hurst parameter*

The problem of Hurst parameter estimation has been widely studied, cf. [6 9]. Our approach is based on self-similarity property of stochastic process $Y = \{Y_t, t \in [t_0, t_T]\}$. If we denote arbitrary selected empirical densities by $\hat{f}_1(Y_{t_1})$ and $\hat{f}_1(Y_{t_k})$, then self-similarity can be described as

$$\hat{f}_1\left(Y_{t_1}\right) \approx k^H \hat{f}_k\left(Y_{t_k}\right). \quad (2)$$

We denote spectral density $f_1(\lambda_j)$ for frequencies $\lambda_j = (2\pi j)/T$, $0 \le j \le [T/2]$. The exact values of spectral densities can be written as

$$f_{C,H}(\lambda_j) = C\lambda^{1-2H}, 0 \le j \le [T/2] \quad (3)$$

where $C$, $H$ are still unknown constant, whose values need to be determined.

So that, in order to estimate $H$ we use the idea periodogram of time series $Y = \{Y_t, t \in [t_0, t_T]\}$.

The covariance coefficient of given time series are

$$\gamma_h = \frac{1}{T-h}\sum_{k=h}^{T-1}\left(Y_{k-h} - \bar{Y}\right)\left(Y_k - \bar{Y}\right), h = 0,1,...,T-1, \quad (4)$$

where $\bar{Y} = \frac{1}{T+1}\sum_{t=0}^{T} Y_t$.

Fourier frequencies are

$$f\left(j,T\right) = \gamma_0 + 2\sum_{h=1}^{T-1}\gamma_h \cos\left(h\lambda_j\right). \quad (5)$$

Now taking into account (2) – (5) we can introduce the functional.

$$\Phi\left(C,H\right) = \sum_j\left[C\lambda_j^{1-2H} - f\left(j,T\right)\right]^2 \xrightarrow[C,H]{} \min \quad (6)$$

Its optimization gives estimate $\hat{H}$ of Hurst parameter $H$.

### *1.3. Task formulation*

We have a stochastic process $Y = \{Y_t, t \in [t_0, t_T]\}$ defined on probability space $(\Omega, F, P)$ and presented by following model

$$dY_t = a(t, Y_t, \psi)dt + b(t, Y_t, \psi)dB_t^H, \quad (7)$$

where $a(t, Y_t, \psi)$, and $b(t, Y_t, \psi)$ are drift and diffusion functions, these both functions satisfy the conditions of existence and uniqueness theorem; $\psi$ is an unknown vector of parameters, which has to be estimated; $B_t^H$ is fBm with Hurst parameter $H$.

## 2. The identification methodology

### 2.1. Propositions

One of the possible approaches of SDEs numerical solution is Monte Carlo method. We will use the idea of this method in order to find estimates of SDE (7), rewriting likelihood functions as

$$\mathsf{L}_\psi^*(Y) = -\ln \mathsf{p}(Y_0, t_0; \psi) - \sum_{i=0}^{T-1} \ln \mathsf{p}(Y_{i+1}, t_{i+1} | Y_i, t_i; \psi), \quad (8)$$

$$\hat{\psi} = \arg\min_\psi \mathsf{L}_\psi^*(Y). \quad (9)$$

The main problem here is how to find estimates $\hat{\mathsf{p}}(Y_{i+1}, t_{i+1} | Y_i, t_i; \psi)$, as far as we have only one observation at any time moment $t_i$ and want to approximate the probability density function in time $t_i$ $(i=1,2,...T)$. The solution of the equation (8) presents a family of $M \in A$ sample paths $Y^j(t_i)$ $(i=1,2,...T, j=1,2,...M)$, thus if we suppose that the estimate of parameter vector $\hat{\psi}$ of SDE (8) is known, then $\mathsf{p}(Y(t_i) \in \{Y^j(t_i)\})=1$ $(j=1,2,...M)$. Now on the basis of values $Y^j(t_i)$ the possibility of the density function estimation for $Y(t_i)$ appears. We can represent (8) as

$$\mathsf{L}_\psi^*(Y) = -\ln \mathsf{f}(Y_0, t_0; \psi) - \sum_{i=1}^{T} \ln \mathsf{f}(Y_i, t_i; \psi), \quad (10)$$

where $\mathsf{f}(Y_i, t_i; \psi)$ is probability density function at time $t_i$ (for convenience we omit $t_i$ and $\psi$).

The probability density function $\mathsf{f}(Y)$ is connected with the probability function $\mathsf{F}(Y)$ as follows

$$\mathsf{f}(Y) = \mathsf{F}'(Y). \quad (11)$$

We use this fact in order to find estimates $\hat{\mathsf{f}}(Y)$. The empirical estimate of probability function is

$$\mathsf{F}(Y) = \frac{1}{M}\sum_{j=1}^{M} 1\{Y^j \le Y\}, \; j=1,2,...,M, \quad (12)$$

where 1 is set belonging indicator.

Using relation (11), $\mathsf{f}(Y)$ can be presented

$$\mathsf{f}(Y) = \lim_{h\to 0}\left(\mathsf{F}(Y+h) - \mathsf{F}(Y)/h\right) \text{, which estimate is}$$

$$\hat{\mathsf{f}}(Y) = \frac{\hat{\mathsf{F}}(Y+h) - \hat{\mathsf{F}}(Y)}{h} = \frac{1}{Mh}\sum_{j=1}^{M} 1\{Y < Y^j \le Y+h\}, \quad (13)$$

where $h$ means the bandwidth, $h$ is absolutely positive constant.

In follows reasoning we will use two operators: operator of mathematical expectation $\mathsf{E}[\cdot]$: and operator of variance $\mathsf{D}[\cdot]$:. $\mathsf{F}(Y)$ is the unbiased estimate of $\hat{\mathsf{F}}(Y)$, i.e. $\mathsf{E}[\hat{\mathsf{F}}(Y)] = \mathsf{F}(Y)$, thus

$$\mathsf{E}\left[\hat{\mathsf{f}}(Y)\right] = \mathsf{f}(Y) = \mathsf{E}\left[\frac{\hat{\mathsf{F}}(Y+h) - \hat{\mathsf{F}}(Y)}{h}\right] - \mathsf{f}(Y) = \quad (14)$$
$$= \frac{1}{Mh}\sum_{j=1}^{M} 1\{Y < Y^j \le Y+h\} - \mathsf{f}(Y)$$

for $h \to 0$ and $M \to \infty$ it is the unbiased estimate too, moreover the variance

$$\mathsf{D}\left[\hat{\mathsf{f}}(Y)\right] = \mathsf{D}\left[\frac{1}{Mh}\sum_{j=1}^{M} 1\{Y < Y^j \le Y+h\}\right] = \quad (15)$$
$$= \frac{1}{Mh^2}\mathsf{D}\left[1\{Y < Y^j \le Y+h\}\right] = \frac{\mathsf{f}(Y)}{Mh} + O\left(\frac{1}{M}\right)$$

is striving to zero for $M \to \infty$ and $Mh \to \infty$. It is clear that quality of the estimate $\mathsf{f}(Y)$ depends on values $M$ and $h$ selection. In order $M \to \infty$, it is enough to increase of $Y^j$ sample paths, thus we will consider problem of $h$ parameter selection.

Let's calculate mathematical expectation of mean squared error of estimate $\hat{\mathsf{f}}(Y)$

$$MSE\left(\mathsf{f}(Y)\right) = \int \mathsf{E}\left[\hat{\mathsf{f}}_h(Y) - \mathsf{f}(Y)\right]^2 dY = \quad (16)$$
$$= \mathsf{E}\int\left[\hat{\mathsf{f}}_h^2(Y) - 2\hat{\mathsf{f}}_h(Y)\mathsf{f}(Y) + \mathsf{f}^2(Y)\right]dY.$$

As far as $h \to 0$ and $Mh \to \infty$, using kernel function [10], expression (16) can be approximated as

$$MSE\left(\hat{\mathsf{f}}(Y)\right) = \frac{1}{Mh}\|K\|_2^2 + \frac{h^4}{4}\left(\mu_2(K)\right)^2 \|\mathsf{f}''(Y)\|_2^2, \quad (17)$$

where $||K||^2$ and $(\mu_2(K))^2$ are some constants depending on kernel function $K$; $\mathsf{f}''(Y)$ is second derivative of function $\mathsf{f}(Y)$.

The minimization of (17) with respect to $h$ gives following results

$$h^{opt} = \left(\frac{1}{M}\frac{\|K\|_2^2}{\|\mathsf{f}''(Y)\|_2^2 \left(\mu_2(K)\right)^2}\right)^{1/5}, \quad (18)$$

where $\|\mathsf{f}''(Y)\|_2^2$ is the only unknown term.

The final solution of (18) depends only on the kernel function solution. For the simplicity we will use for SDE (8) parametric identification Epanichnikov kernel function [11]:

$$K(u) = \frac{3}{4}\left(1-u^2\right)1\left(|u| \le 1\right), \quad (19)$$

where $u = (Y^j - Y)/h$.

Substituting (19) into (18) we get

$$h^{opt} \approx 0.9\sigma_Y M^{-1/5}, \quad (20)$$

where is standard deviation of the sample $Y^j$ $(j=1,2,...,M)$.

### 2.2. Algorithm

Now function (10) can be estimated by the algorithm:

1. Using numerical method for SDE solution generate $M$ realizations of the process for $t_i\,(j=1,2,...,T)$, getting sequence of $Y^j(t_i)$, $(j=1,2,...,M)$.
2. For $M$ observation $Y^j(t_i)$, $(j=1,2,..., M)$, of random value $Y^j(t_i)$ estimate values of probability density function $f(Y(t_i))$

$$\hat{f}(Y(t_i))=\frac{3}{4Mh}\sum_{j=1}^{M}\left(1-\left(\frac{Y(t_i)-Y^j(t_i)}{h}\right)^2\right)1\{Y(t_i)- \tag{21}$$

$$-h\le Y^j(t_i)\le Y(t_i)+h\},$$

3. Steps 1) and 2) have to be repeat for all $T$ and substitute results into (10).

As far as $h\to 0$, but $h\neq 0$, then estimates (21) are biased, thus $E\left[\ln\hat{f}(Y(t_i))\right]-\ln f(Y(t_i))\neq 0$, thus final estimate of (10) has to be corrected at any point $t_i$ as

$$\ln f(Y(t_i))=\ln\left(E\left[\hat{f}(Y(t_i))\right]\right)-\frac{1}{2}\frac{D\left[\hat{f}(Y(t_i))\right]}{\left(E\left[\hat{f}(Y(t_i))\right]\right)^2}, \tag{22}$$

where

$$E\left[\hat{f}(Y(t_i))\right]=\frac{1}{m_2}\sum_{m=1}^{m_2}\hat{f}_m(Y(t_i)) \tag{23}$$

and

$$D\left[\hat{f}(Y(t_i))\right]=\frac{1}{m_2-1}\sum_{m=1}^{m_2}\left(\hat{f}_{m_2}(Y(t_i))-E\left[\hat{f}(Y(t_i))\right]\right)^2. \tag{24}$$

Numerical solution of (3.1) can be completed by Euler scheme, where pseudorandom numbers have to be generated by some suitable algorithm. We will discuss one of algorithms in next section.

## 3. Simulation results

### 3.1. Pseudorandom numbers generator

Mandelbrot and van Ness presented fBm by its stochastic representation with respect to ordinary Brownian motion [5]

$$B_t^H=\frac{1}{\Gamma(H+1/2)}\left(\int_{-\infty}^{0}\left[(t-s)^{H-1/2}-(-s)^{H-1}\right]dB(s)+\right. \tag{25}$$

$$\left.+\int_0^t\left[(t-s)^{H-1/2}\right]dB(s)\right),\, s<t,$$

where $\Gamma(\cdot)$ represents the Gamma function.

The idea of stochastic representation method is to approximate this integral by Riemann type sums to simulate the process $\tilde{B}_t^H\,(t=0,1,...,T)$:

$$\tilde{B}_t^H=C^H\left(\sum_{k=-d}^{0}\left[(t-k)^{H-1/2}-(-k)^{H-1}\right]B_1(k)+\right. \tag{26}$$

$$\left.+\sum_{k=0}^{t}(t-k)^{H-1/2}B_2(k)\right),$$

where $B_1$ and $B_2$ are vectors of $d+1$ and $T+1$ $(d\neq T)$ are observation of white noise and

$$C^H=\sqrt{\frac{\Gamma(2H+1)\sin(\pi H)}{\Gamma(H+1/2)}}. \tag{27}$$

The illustration of this generation method for $T=1000$ and $H=\{0.2{:}0.5{:}0.7\}$ is presented in Fig.1.

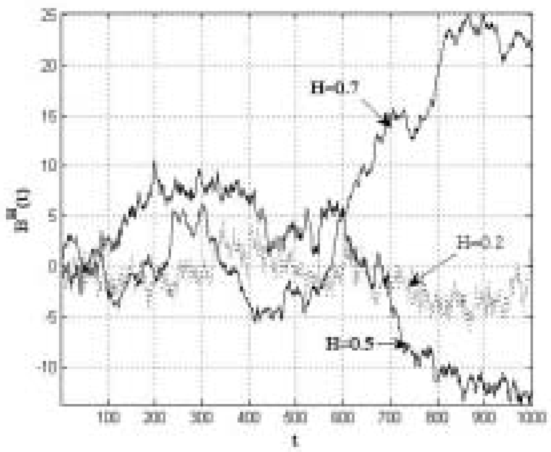


*Figure 1. Sample paths of fBm with different Hurst parameters values simulated by Mandelbrot and van Ness method.*

### 3.2. Numerical example

In order to check properties of simulated maximum likelihood estimation approach we will complete some numerical experiments using Monte Carlo simulations. All the experiments consist from:

1) $B^H=\{B_t^H, t\in[t_0,t_T]\}$ sample paths generation using Mandelbrot and van Ness method;
2) Hurst parameter estimation by semiparametric method;
3) SDE sample paths generation by Euler numerical scheme:

$$Y_{t+1}=Y_t+a(t,Y_t,\psi)\Delta_t+b(t,Y_t,\psi)\Delta B_t^H, \tag{28}$$

where $\Delta_t=[t_T-t_0]/T$, $\Delta B_t^H=B_{t+1}^H-B_t^H$;

4) SDE parameters estimation by simulated likelihood method.

During experiments we examine only linear version of SDE (7) as far as in fBm case Lebesgue-Stieltjes integration and Ito stochastic calculus cannot be applied, creation numerical schemes with better convergence is not so convenient. Using subscripts to denote the time index, the test equations are

$$dY_t=\psi_1 Y_t dt+\psi_2 Y_t dB_t^H, \tag{A}$$

$$dY_t=Y_t dt+\psi_2^2 dB_t^H, \tag{B}$$

$$dY_t=\frac{1}{2}\psi_1^2 Y_t dt+\psi_2 Y_t dB_t^H, \tag{C}$$

$$dY_t=\frac{1}{\psi_1}Y_t dt+\psi_2 Y_t dB_t^H, \tag{D}$$

$$dY_t=\psi_1 Y_t dt+\frac{1}{2\psi_2}Y_t\, dB_t^H. \tag{E}$$

Estimation of the drift and diffusion parameters of equations (A) – (E) is conducted for different values of Hurst parameter $H=\{0.2;0,5;0.9\}$. Time interval $[t_0, t_T]$ is divided on $T=1000$ subintervals, where $t_0=0$ and $t_T=1$. For SDE sample paths $Y_t^j$, $(j=1,...,1000)$ generation is used the initial value $Y_0=0.5$ and true parameters $\psi_1=1.0$ and $\psi_2=0.5$. Final estimate of (5) is to be corrected at any instant, where $m_2=250$. The derivative-free simplex method [12] is used to estimate $\psi_1$ and $\psi_2$ in (2b) using start values of $\hat{\psi}_1=1.3$ and $\hat{\psi}_2=0.4$, initial value $\hat{Y}_0=0.1$. The mean and standard deviation of parameters $\psi_1$ and $\psi_2$ estimates are reported in Table I.

Analysis of obtained results shows that for all test equations with $H=0.3$ the absolute error of $\psi_1$ and $\psi_2$ in some cases exceeds even 15%. The relatively small values of $s_{\psi1}$ and $s_{\psi2}$ demonstrate significant right-hand side bias of estimates mean values. Study of white noise case for all test equations illustrates that the absolute error does not exceed 2.5%, as well as the low values of $s_{\psi1}$ and $s_{\psi2}$ allow to conclude that mean values $\hat{\psi}_1$ and $\hat{\psi}_2$ do not have significant bias. In the case of long-range depended process, where $H=0.8$, for all equations the absolute error for $\hat{\psi}_1$ is negligible, but for $\hat{\psi}_2$ in some cases exceeds 10% and has left-side bias.

## 4. Concluding remarks

In this paper, we proposed the methodology of parameter estimation in SDE driven by fBm. The underlying idea of this methodology is based on the SDE maximum likelihood function simulation. For this purpose we used wavelet-based generation methods of fBm sample paths and semiparametric method of Hurst parameter estimation. Numerical study demonstrated the effectiveness of the methodology despite on the high values of absolute error for some SDE parameters estimates. To improve the methodology, we may consider using different schemes of SDE numerical solution of higher order of convergence.

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*Table 1. Results of Monte Carlo simulation.*

| SDE | Hurst parameter, $H$ | Estimate of mean value | | Estimate of standard deviation | |
|---|---|---|---|---|---|
| | | $\hat{\psi}_1$ | $\hat{\psi}_2$ | $s_{\psi1}$ | $s_{\psi2}$ |
| (A) | 0.3 | 1.1189 | 0.5624 | 0.1335 | 0.0524 |
| | 0.5 | 1.0152 | 0.5192 | 0.0313 | 0.0299 |
| | 0.8 | 0.9975 | 0.4193 | 0.0219 | 0.0687 |
| (B) | 0.3 | – | 0.5589 | – | 0.0611 |
| | 0.5 | – | 0.5142 | – | 0.0315 |
| | 0.8 | – | 0.5009 | – | 0.0701 |
| (C) | 0.3 | 1.2552 | 0.5853 | 0.1447 | 0.0471 |
| | 0.5 | 1.0102 | 0.5133 | 0.0401 | 0.0301 |
| | 0.8 | 0.9877 | 0.4452 | 0.0219 | 0.0687 |
| (D) | 0.3 | 0.9106 | 0.5055 | 0.3103 | 0.0346 |
| | 0.5 | 1.0118 | 0.5006 | 0.0397 | 0.0225 |
| | 0.8 | 0.9975 | 0.4193 | 0.0219 | 0.0687 |
| (E) | 0.3 | 1.2418 | 0.5309 | 0.1455 | 0.0149 |
| | 0.5 | 1.0298 | 0.5069 | 0.0455 | 0.0125 |
| | 0.8 | 1.0031 | 0.4524 | 0.0305 | 0.0591 |

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