STATE OF THE ART

State Variables Diagram Method for Determination of Positive Realizations of 2D Systems with Delays

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Abstract:
Realization problem for positive single-input single-output systems with delays in state vector and inputs described by the 2D general model is addressed. Sufficient condition for the existence of a positive realization are established and a procedure for finding a positive realization for a given proper transfer function is proposed. The procedure is illustrated by a numerical example.

Keywords: *positive realization, existence, procedure, 2D general model, delay*

1. Introduction

The most popular models of two–dimensional (2D) linear systems are the models introduced by Roesser [22], Fornasini-Marchesini [4,5] and Kurek [22]. The models have been extended for positive systems in [10, 24, 13, 9]. An overview of 2D linear system theory is given in [1, 2, 7, 8] and some recent result in positive systems has been given in the monographs [3,9] and in paper [12,24]. Reachability and minimum energy control of positive 2D systems with one delay in states have been considered in [13] and the upper bound for the reachability index of the positive 2D general model has been analyzed in [11].

The notion of internally positive 2D system (model) with delays in states and in inputs has been introduced and necessary and sufficient conditions for the internal positivity, reachability, controllability, observability and the minimum energy control problem have been established in [18].

The realization problem for 1D positive discrete-time and continuous-time systems with delays has been analyzed in [14-17, 19, 20]. The realization problem for 2D standard systems has been investigated in many papers and books [4, 5, 6, 7, 8] and for 2D positive systems with output delays in [9].

In this paper sufficient conditions will be established for the existence of a positive realization for 2D linear systems with delays and a procedure will be proposed for finding a positive realization for a given proper transfer function.

To the best knowledge of the author the problem for positive 2D systems with delays in state and in inputs have not been considered yet.

2. Conception of construction

Let R_+^{nxm} be the set of nxm real matrices with nonnegative entries and $R_+^n = R_+^{nx1}$. The set of integers nonnegative will be denoted by Z_+ and the nxm identity matrix by $\boldsymbol{I}_n$.

Consider the 2D single-input singleoutput system (model) with delays in state and in input

$$x(i+1, j+1) = \sum\sum_{k,l\in D_{n_1n_2}} \left(\mathbf{A}_{kl}x(i-k, j-l) + \mathbf{B}_{kl}u(i-k, j-l)\right) \quad (1a)$$

$$y(i,j) = \mathbf{C}x(i,j) + \mathbf{D}u(i,j), \quad i,j\in Z_+ \quad (1b)$$

where $x(i,j)\in R^n$, $u(i,j)\in R^n$, $y(i,j)\in R^n$ are the state vector, input and output, respectively and, $\boldsymbol{A}_{kl}\in R^{nxn}$, $\boldsymbol{B}_{kl}\in R^n$, $k,l\in D_{n_1n_2}$, $\boldsymbol{C}\in R^{1xn}$, $\boldsymbol{D}\in R$

$$D_{n_1n_2} = \{k,l\in Z \;:\; -1\le k\le n_1,\; -1\le l\le n_2,\; k+l > -2\}$$

Definition 1. The system (1) is called (internally) positive if for all boundary conditions

$$x(i-k,-l)\in R_+^n, \quad x(-k, j-l)\in R_+^n,$$
$$k = 0,1,\dots,n_1;\; l = 0,1,\dots n_2;\; i,j\in Z_+$$

and every input sequence $u(i,j)\in R_+$, $i,j\in D_{n_1n_2}$ we have $x(i,j)\in R_+^n$ and $y(i,j)\in R_+$ for $i,j\in Z_+$.

Theorem 1 [18]. The system (1) is (internally) positive if and only if

$$\mathbf{A}_{kl}\in R_+^{n\times n},\; \mathbf{B}_{kl}\in R_+^n,\; k,l\in D_{n_1n_2},\; \mathbf{C}\in R_+^{1\times n},\; D\in R_+ \quad (3)$$

The transfer function of (1) is given by

$$T(z_1,z_2) = \mathbf{C}\left[I_n z_1 z_2 - \sum\sum_{k,l\in D_{n_1n_2}} \mathbf{A}_{kl}z_1^{-k}z_2^{-l}\right]^{-1}\left(\sum\sum_{k,l\in D_{n_1n_2}} \mathbf{B}_{kl}z_1^{-k}z_z^{-l}\right) + D \quad (4)$$

Definition 2. Matrices (3) are called a positive realization of a given transfer function $T(z_1,z_2)$ if they satisfy the equality (4).

The realization problem can be stated as follows. Given a proper transfer function $T(z_1,z_2)$, find a positive realization (3) of $T(z_1,z_2)$.

In this paper sufficient conditions for the existence of a positive realization of a given $T(z_1,z_2)$ will be established and a procedure for finding of a positive realization (3) of $T(z_1,z_2)$ will be proposed.

3. Problem solution

From (4) we have

$$D = \lim_{\substack{z_1\to\infty \\ z_2\to\infty}} T(z_1, z_2) \tag{5}$$

since

$$\lim_{\substack{z_1\to\infty \\ z_2\to\infty}} \left[\mathbf{I}_n z_1 z_2 - \sum\sum_{k,l\in D_{n_1 n_2}} \mathbf{A}_{kl} z_1^{-k} z_2^{-l} \right]^{-1} = 0 \tag{6}$$

The strictly proper part of $T(z_1,z_2)$ is given by

$$T_{sp}(z_1, z_2) = T(z_1, z_2) - D = \frac{n(z_1, z_2)}{d(z_1, z_2)} \tag{7}$$

where

$$n(z_1, z_2) = \mathbf{C}\ \mathrm{Adj} \left[\mathbf{I}_n z_1 z_2 - \sum\sum_{k,l\in D_{n_1 n_2}} \mathbf{A}_{kl} z_1^{-k} z_2^{-l} \right] \left(\sum\sum_{k,l\in D_{n_1 n_2}} \mathbf{B}_{kl} z_1^{-k} z_2^{-l} \right) \tag{8}$$

$$d(z_1, z_2) = \det \left[\mathbf{I}_n z_1 z_2 - \sum\sum_{k,l\in D_{n_1 n_2}} \mathbf{A}_{kl} z_1^{-k} z_2^{-l} \right] \tag{9}$$

Therefore, the positive realization problem has been reduced to finding the matrices

$$\mathbf{A}_{kl} \in \mathrm{R}_+^{n\times n},\ \ \mathbf{B}_{kl} \in \mathrm{R}_+^{n},\ \ k,l\in D_{n_1 n_2},\ \text{ and } \ \mathbf{C} \in \mathrm{R}_+^{1\times n} \tag{10}$$

for a given strictly proper transfer function (7).

Let a given irreducible transfer function have the form

$$T(z_1, z_2) = \frac{\sum_{k=0}^{N}\sum_{l=0}^{M} \bar{b}_{kl} z_1^k z_2^l}{z_1^N z_2^M - \sum_{\substack{k=0 \\ k+l<N+M}}^{N}\sum_{l=0}^{M} a_{kl} z_1^k z_2^l} \tag{11}$$

Using (5) for (11) we obtain

$$D = \lim_{\substack{z_1\to\infty \\ z_2\to\infty}} T(z_1, z_2) = \bar{b}_{NM} \tag{12}$$

and

$$T_{sp}(z_1, z_2) = T(z_1, z_2) - \bar{b}_{NM} = \frac{\sum_{\substack{k=0 \\ k+l<N+M}}^{N}\sum_{l=0}^{M} b_{kl} z_1^k z_z^l}{z_1^N z_2^M - \sum_{\substack{k=0 \\ k+l<N+M}}^{N}\sum_{l=0}^{M} a_{kl} z_1^k z_z^l} \tag{13}$$

where

$$b_{kl} = \bar{b}_{kl} + a_{kl}\bar{b}_{NM} \qquad k = 0,1,\ldots,N;\ l = 0,1,\ldots.,M,\ \ k+l < N+M \tag{13a}$$

$$T_{sp}(z, w) = \frac{\sum_{\substack{k=0 \\ k+l<N+M}}^{N}\sum_{l=0}^{M} b_{N-k,M-l} z^k w^l}{1 - \sum_{\substack{k=0 \\ k+l<N+M}}^{N}\sum_{l=0}^{M} a_{N-k,M-l} z^k w^l} \tag{14}$$

where $z = z_1^{-1}$ and $w = z_2^{-1}$.

Let Y and U be the 2D Z transforms of $y(i,j)$ and $u(i,j)$. Taking into account that $T_{sp}(z,w) = Y/U$ and using (14) we may write

$$Y = \sum_{\substack{k=0 \\ k+l>0}}^{N}\sum_{l=0}^{M} b_{N-k,M-l} z^k w^l U + \sum_{\substack{k=0 \\ k+l>0}}^{N}\sum_{l=0}^{M} a_{N-k,M-l} z^k w^l Y$$

and

$$\begin{aligned} Y = {} & \sum_{l=1}^{M} b_{N,M-l} w^l U + \sum_{l=1}^{M} a_{N,M-l} w^l Y + z\Bigg(\sum_{l=0}^{M} b_{N-1,M-l} w^l U + \\ & + \sum_{l=0}^{M} a_{N-1,M-l} w^l Y + z\Bigg(\sum_{l=0}^{M} b_{N-2,M-l} w^l U + \sum_{l=0}^{M} a_{N-2,M-l} w^l Y + \mathrm{K} + \\ & + z\left(\sum_{l=0}^{M} b_{0,M-l} w^l U + \sum_{l=0}^{M} a_{0,M-l} w^l Y \right) \mathrm{K} \Bigg)\Bigg) \end{aligned} \tag{15}$$

From (15) it follows the state variable diagram shown in Fig. 1 for $N = 6$ and $M = 4$. The number of horizontal delay elements (denoted by z) is equal to N and the number of vertical delay elements (denoted by w) is equal to M.

Case 1: $N = nn_1$, $M = n_2$ (or $N = n_1$, $M = nn_2$)
Case 2: $N = nn_1$, $M = nn_2$.

First the essence of the proposed method will be presented for $N = 6$ and $M = 4$.

Case 1. W choose $n = 3$, $n_1 = 2$ and $n_2 = 4$. As the state variables $x_1(i,j)$, $x_2(i,j)$ and $x_3(i,j)$ we choose the outputs of the second, fourth and sixth horizontal delay elements. Using the state variable diagram shown in Fig. 1 we may write the following equations:

$$\begin{aligned} x_1(i+1, j+1) = {} & a_{14}x_3(i, j+1) + a_{13}x_3(i,j) + a_{12}x_3(i, j-1) + a_{11}x_3(i, j-2) + a_{10}x_3(i, j-3) + a_{04}x_3(i-1, j+1) + a_{03}x_3(i-1, j) + \\ & + a_{02}x_3(i-1, j-1) + a_{01}x_3(i-1, j-2) + a_{00}x_3(i-1, j-3) + b_{14}u(i, j+1) + b_{13}u(i,j) + b_{12}u(i, j-1) + b_{11}u(i, 2-j) + \\ & + b_{10}u(i, j-3) + b_{04}u(i-1, j+1) + b_{03}u(i-1, j) + b_{02}u(i-1, j-1) + b_{01}u(i-1, j-2) + b_{00}u(i-1, j-3) \end{aligned} \tag{16}$$

$$\begin{aligned} x_2(i+1, j+1) = {} & a_{34}x_3(i, j+1) + a_{33}x_3(i,j) + a_{32}x_3(i, j-1) + a_{31}x_3(i, j-2) + a_{30}x_3(i, j-3) + a_{24}x_3(i-1, j+1) + a_{23}x_3(i-1, j) + \\ & + a_{22}x_3(i-1, j-1) + a_{21}x_3(i-1, j-2) + a_{20}x_3(i-1, j-3) + x_1(i-1, j+1) + b_{34}u(i, j+1) + b_{33}u(i,j) + b_{32}u(i, j-1) + \\ & + b_{31}u(i, j-2) + b_{30}u(i, j-3) + b_{24}u(i-1, j+1) + b_{23}u(i-1, j) + b_{22}u(i-1, j-1) + b_{21}u(i-1, j-2) + b_{20}u(i-1, j-3) \end{aligned}$$

$$\begin{aligned} x_3(i+1, j+1) = {} & a_{63}x_3(i+1, j) + a_{62}x_3(i+1, j-1) + a_{61}x_3(i+1, j-2) + a_{60}x_3(i+1, j-3) + x_2(i-1, j+1) + a_{54}x_3(i, j+1) + a_{53}x_3(i,j) + \\ & + a_{52}x_3(i, j-1) + a_{51}x_3(i, j-2) + a_{50}x_3(i, j-3) + a_{44}x_3(i-1, j+1) + a_{43}x_3(i-1, j) + a_{42}x_3(i-1, j-1) + a_{41}x_3(i-1, j-2) + \\ & + a_{40}x_3(i-1, j-3) + b_{63}u(i+1, j) + b_{62}u(i+1, j-1) + b_{61}u(i+1, j-2) + b_{60}u(i+1, j-3) + b_{54}u(i, j+1) + b_{53}u(i,j) + \\ & + b_{52}u(i, j-1) + b_{51}u(i, j-2) + b_{50}u(i, j-3) + b_{44}u(i-1, j+1) + b_{43}u(i-1, j) + b_{42}u(i-1, j-1) + b_{41}u(i-1, j-2) + b_{40}u(i-1, j-3) \end{aligned}$$

The equation (16) can be written in the form

$$\begin{aligned} x(i+1,j+1) = {} & A_{-1,0}x(i+1,j) + A_{-1,1}x(i+1,j-1) + A_{-1,2}x(i+1,j-2) + A_{-1,3}x(i+1,j-3) + A_{0,-1}x(i,j+1) + A_{0,0}x(i,j) + A_{0,1}x(i,j-1) + \\ & + A_{0,2}x(i,j-2) + A_{0,3}x(i,j-3) + A_{1,-1}x(i-1,j+1) + A_{1,0}x(i-1,j) + A_{1,1}x(i-1,j-1) + A_{1,2}x(i-1,2-j) + A_{1,3}x(i-1,j-3) + \\ & + B_{-1,0}u(i+1,j) + B_{-1,1}u(i+1,j-1) + B_{-1,2}u(i+1,j-2) + B_{-1,3}u(i+1,j-3) + B_{0,-1}u(i,j+1) + B_{0,0}u(i,j) + B_{0,1}u(i,j-1) + \\ & + B_{0,2}u(i,i-2) + B_{0,3}u(i,j-3) + B_{1,-1}u(i-1,j+1) + B_{1,0}u(i-1,j) + B_{1,1}u(i-1,j-1) + B_{1,2}u(i-1,j-2) + B_{1,3}u(i-1,j-3) \end{aligned} \quad (17)$$

where

$$\mathbf{A}_{-1,0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{63} \end{bmatrix}, \quad \mathbf{A}_{-1,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{62} \end{bmatrix}, \quad \mathbf{A}_{-1,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{61} \end{bmatrix}, \quad \mathbf{A}_{-1,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{60} \end{bmatrix}$$

$$\mathbf{A}_{0,-1} = \begin{bmatrix} 0 & 0 & a_{14} \\ 0 & 0 & a_{34} \\ 0 & 0 & a_{54} \end{bmatrix}, \quad \mathbf{A}_{0,0} = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{33} \\ 0 & 0 & a_{53} \end{bmatrix}, \quad \mathbf{A}_{0,1} = \begin{bmatrix} 0 & 0 & a_{12} \\ 0 & 0 & a_{32} \\ 0 & 0 & a_{52} \end{bmatrix}, \quad \mathbf{A}_{0,2} = \begin{bmatrix} 0 & 0 & a_{11} \\ 0 & 0 & a_{31} \\ 0 & 0 & a_{51} \end{bmatrix},$$

$$\mathbf{A}_{0,3} = \begin{bmatrix} 0 & 0 & a_{10} \\ 0 & 0 & a_{30} \\ 0 & 0 & a_{50} \end{bmatrix}, \quad \mathbf{A}_{1,-1} = \begin{bmatrix} 0 & 0 & a_{04} \\ 1 & 0 & a_{24} \\ 0 & 1 & a_{44} \end{bmatrix}, \quad \mathbf{A}_{1,0} = \begin{bmatrix} 0 & 0 & a_{03} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{43} \end{bmatrix}, \quad \mathbf{A}_{1,1} = \begin{bmatrix} 0 & 0 & a_{02} \\ 0 & 0 & a_{22} \\ 0 & 0 & a_{42} \end{bmatrix},$$

$$\mathbf{A}_{1,2} = \begin{bmatrix} 0 & 0 & a_{01} \\ 0 & 0 & a_{21} \\ 0 & 0 & a_{41} \end{bmatrix}, \quad \mathbf{A}_{1,3} = \begin{bmatrix} 0 & 0 & a_{00} \\ 0 & 0 & a_{20} \\ 0 & 0 & a_{40} \end{bmatrix}, \quad \mathbf{B}_{-1,0} = \begin{bmatrix} 0 \\ 0 \\ b_{63} \end{bmatrix}, \quad \mathbf{B}_{-1,1} = \begin{bmatrix} 0 \\ 0 \\ b_{62} \end{bmatrix}, \quad \mathbf{B}_{-1,2} = \begin{bmatrix} 0 \\ 0 \\ b_{61} \end{bmatrix},$$

$$\mathbf{B}_{-1,3} = \begin{bmatrix} 0 \\ 0 \\ b_{60} \end{bmatrix}, \quad \mathbf{B}_{0,-1} = \begin{bmatrix} b_{14} \\ b_{34} \\ b_{54} \end{bmatrix}, \quad \mathbf{B}_{0,0} = \begin{bmatrix} b_{13} \\ b_{33} \\ b_{53} \end{bmatrix}, \quad \mathbf{B}_{0,1} = \begin{bmatrix} b_{12} \\ b_{32} \\ b_{52} \end{bmatrix}, \quad \mathbf{B}_{0,2} = \begin{bmatrix} b_{11} \\ b_{31} \\ b_{51} \end{bmatrix}, \quad \mathbf{B}_{0,3} = \begin{bmatrix} b_{10} \\ b_{30} \\ b_{50} \end{bmatrix},$$

$$\mathbf{B}_{1,-1} = \begin{bmatrix} b_{04} \\ b_{24} \\ b_{44} \end{bmatrix}, \quad \mathbf{B}_{1,0} = \begin{bmatrix} b_{03} \\ b_{23} \\ b_{43} \end{bmatrix}, \quad \mathbf{B}_{1,1} = \begin{bmatrix} b_{02} \\ b_{22} \\ b_{42} \end{bmatrix}, \quad \mathbf{B}_{1,2} = \begin{bmatrix} b_{01} \\ b_{21} \\ b_{41} \end{bmatrix}, \quad \mathbf{B}_{1,3} = \begin{bmatrix} b_{00} \\ b_{20} \\ b_{40} \end{bmatrix}. \quad (18)$$

From the output equation

$$y(i,j) = x_3(i,j) \quad (19)$$

we have

$$\mathbf{C} = [0 \quad 0 \quad 1] \quad (20)$$

In a similar way in general case we obtain matrices $\boldsymbol{A}_{kl} \in R^{nxm}$ of the form

$$\mathbf{A}_{-1,0} = \begin{bmatrix} 0 & K & 0 & 0 \\ M & O & M & M \\ 0 & K & 0 & 0 \\ 0 & K & 0 & a_{nn_1,n_2-1} \end{bmatrix}, \quad \mathbf{A}_{-1,1} = \begin{bmatrix} 0 & K & 0 & 0 \\ M & O & M & M \\ 0 & K & 0 & 0 \\ 0 & K & 0 & a_{nn_1,n_2-2} \end{bmatrix}, K, \mathbf{A}_{-1,n_2-1} = \begin{bmatrix} 0 & K & 0 & 0 \\ M & O & M & M \\ 0 & K & 0 & 0 \\ 0 & K & 0 & a_{nn_1,0} \end{bmatrix}$$

$$\mathbf{A}_{0,-1} = \begin{bmatrix} 0 & K & 0 & a_{n_1-1,n_2} \\ 0 & K & 0 & a_{2n_1-1,n_2} \\ M & O & M & M \\ 0 & K & 0 & a_{nn_1-1,n_2} \end{bmatrix}, \quad \mathbf{A}_{0,0} = \begin{bmatrix} 0 & K & 0 & a_{n_1-1,n_2-1} \\ 0 & K & 0 & a_{2n_1-1,n_2-1} \\ M & O & M & M \\ 0 & K & 0 & a_{nn_1-1,n_2-1} \end{bmatrix}, K, \mathbf{A}_{0,n_2-1} = \begin{bmatrix} 0 & K & 0 & a_{n_1-1,0} \\ 0 & K & 0 & a_{2n_1-1,0} \\ M & O & M & M \\ 0 & K & 0 & a_{nn_1-1,0} \end{bmatrix}$$

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$$\mathbf{A}_{n_1-1,-1} = \begin{bmatrix} 0 & 0 & K & a_{0,n_2} \\ 0 & 1 & K & a_{n_1,n_2} \\ M & M & O & M \\ 0 & K & 1 & a_{(n-1)n_1,n_2} \end{bmatrix}, \mathbf{A}_{n_1-1,0} = \begin{bmatrix} 0 & K & 0 & a_{0,n_2-1} \\ 0 & K & 0 & a_{n_1,n_2-1} \\ M & O & M & M \\ 0 & K & 0 & a_{(n-1)n_1,n_2-1} \end{bmatrix}, K$$

$$K, \mathbf{A}_{n_1-1,n_2-1} = \begin{bmatrix} 0 & 0 & K & a_{00} \\ 0 & 1 & K & a_{n_1,0} \\ M & M & O & M \\ 0 & K & 0 & a_{(n-1)n_1,0} \end{bmatrix} \quad (21)$$

and the matrices $\mathbf{B}_{kl} \in R^n$ of the form

$$\mathbf{B}_{-1,0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{nn_1,n_2-1} \end{bmatrix}, \quad \mathbf{B}_{-1,1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{nn_1,n_2-2} \end{bmatrix}, \ldots, \mathbf{B}_{-1,n_2-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{nn_1,0} \end{bmatrix} \tag{22}$$

$$\mathbf{B}_{0,-1} = \begin{bmatrix} b_{n_1-1,n_2} \\ b_{2n_1-1,n_2} \\ \vdots \\ b_{nn_1-1,n_2} \end{bmatrix}, \quad \mathbf{B}_{0,0} = \begin{bmatrix} b_{n_1-1,n_2-1} \\ b_{2n_1-1,n_2-1} \\ \vdots \\ b_{nn_1-1,n_2-1} \end{bmatrix}, \ldots, \mathbf{B}_{0,n_2-1} = \begin{bmatrix} b_{n_1-1,0} \\ b_{2n_1-1,0} \\ \vdots \\ b_{nn_1-1,0} \end{bmatrix}$$

....

$$\mathbf{B}_{n_1-1,-1} = \begin{bmatrix} b_{0,n_2} \\ b_{n_1,n_2} \\ \vdots \\ b_{(n-1)n_1,n_2} \end{bmatrix}, \quad \mathbf{B}_{n_1-1,0} = \begin{bmatrix} b_{0,n_2-1} \\ b_{n_1,n_2-1} \\ \vdots \\ b_{(n-1)_2 n_1,n_2-1} \end{bmatrix}, \ldots, \mathbf{B}_{n_1-1,n_2-1} = \begin{bmatrix} b_{00} \\ b_{n_1,0} \\ \vdots \\ b_{(n-1)n_1,0} \end{bmatrix}$$

The matrix $\mathbf{C}$ has the form

$$\mathbf{C} = [0 \quad \ldots \quad 0 \quad 1] \in \mathrm{R}^{1\times n} \tag{23}$$

Theorem 2. There exists a positive realization (3) of the transfer function (11) for $N = nn_1$ and $M = n_2$ if the coefficients a_{kl} and $\bar{b}_{kl}$ are nonnegative

$$a_{kl} \geq 0 \quad and \quad \bar{b}_{kl} \geq 0 \quad for \quad k = 0,1,...,N; \; l = 0,1,...,M \tag{24}$$

Proof. From (12) it follows that $b_{NM} \geq 0$ implies $\mathbf{D} \in \mathrm{R}_+$. If the conditions (24) are satisfied then from (13a) we have $b_{kl} \geq 0$ for $k=0,1,\ldots,N$; $l=0,1,\ldots,M$, $k+l<N+M$ and this implies the nonnegativity of the matrices (22). The condition $a_{kl} \geq 0$ $k=0, 1,\ldots, N$; $l=0, 1,\ldots, M$ implies the nonnegativity of the matrices (21). From (23) we have $\mathbf{C} \in \mathrm{R}_+^{1\times N}$.

If the conditions (24) are met, then a positive realization (3) of the transfer function (11) for $N = nn_1$ and $M = n_2$ can be found by the use the following procedure.

Procedure 1.

Step 1: Knowing the degrees N and M of the denominator $d(z_1,z_2)$ of (11) choose n, n_1 and n_2 such that $nn_1 = N$ and $n_2 = M$.

Step 2: Using (5) and (7) find $\mathbf{D}$ and the strictly proper transfer function (7).

Step 3: Using (21), (22) and (23) find the matrices $\mathbf{A}_{kl}$, $\mathbf{B}_{kl}$ for $k = -1,0,\ldots,n_1-1$; $l = 0,1,\ldots,n_2-1$ and $\mathbf{C}$.

Remark 1. Note that the role of the variables z and w (z_1 and z_2) in the above considerations can be interchanged.

Example 1. Find a positive realization (3) of the transfer function

$$T(z_1,z_2) = \frac{2z_1^4 z_2^2 + z_1^3 z_2^2 + z_1 z_2 + z_1^2 + 1}{z_1^4 z_2^2 - 2z_1^3 z_2^2 - z_1^2 z_2 - z_1 z_2 - z_1^2 - z_2 - 1} \tag{25}$$

The transfer function (25) satisfies condition (24). Using Procedure 1 we obtain.

Step 1. In his case N=4 and M=2 and we choose n=2 and n_1=n_2=2

Step 2. Using (5) and (7) we obtain

$$D = \lim_{\substack{z_1\to\infty \\ z_2\to\infty}} T(z_1,z_2) = 2 \tag{26}$$

and

$$T_{sp}(z_1,z_2) = T(z_1,z_2) - D = \frac{5z_1^3 z_2^2 + 2z_1^2 z_2 + 3z_1 z_2 + 3z_1^2 + 2z_2 + 3}{z_1^4 z_2^2 - 2z_1^3 z_2^2 - z_1^2 z_2 - z_1 z_2 - z_1^2 - z_2 - 1} \tag{27}$$

Multiplying the numerator and denominator of (27) by $z_1^{-4} z_2^{-2}$ we obtain

$$T_{sp}(z,w) = \frac{5z + 2z^2 w + 3z^3 w + 3z^2 w^2 + 2z^4 w + 3z^4 w^2}{1 - 2z - z^2 w - z^3 w - z^2 w^2 - z^4 w - z^4 w^2} \tag{28}$$

Step 3. Taking into account that

$$a_{00} = a_{01} = a_{11} = a_{20} = a_{21} = 1, \; a_{32} = 2$$
$$b_{00} = b_{20} = b_{11} = 3, \; b_{21} = b_{01} = 2, \; b_{32} = 5 \quad \text{(remaining coefficients are zero)}$$

and using (21), (22) and (23) we obtain (zero matrices are omitted):

$$\mathbf{A}_{0,-1} = \begin{bmatrix} 0 & a_{12} \\ 0 & a_{32} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \; \mathbf{A}_{00} = \begin{bmatrix} 0 & a_{11} \\ 0 & a_{31} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{29}$$
$$\mathbf{A}_{1,-1} = \begin{bmatrix} 0 & a_{02} \\ 1 & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; \mathbf{A}_{1,0} = \begin{bmatrix} 0 & a_{01} \\ 0 & a_{21} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$
$$\mathbf{A}_{1,1} = \begin{bmatrix} 0 & a_{00} \\ 0 & a_{20} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \; \mathbf{B}_{0,-1} = \begin{bmatrix} b_{12} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \; \mathbf{B}_{0,0} = \begin{bmatrix} b_{11} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$
$$\mathbf{B}_{1,0} = \begin{bmatrix} b_{01} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \; \mathbf{B}_{1,1} = \begin{bmatrix} b_{00} \\ b_{20} \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \; \mathbf{C} = [0 \quad 1]$$

The desired positive realization of (25) is given by (26) and (29) and the 2D systems is described by equations

$$\begin{aligned} x(i+1,j+1) = &\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} x(i,j+1) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(i,j) + \\ &+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(i+1,j-1) + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x(i-1,j) + \\ &+ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x(i-1,j-1) + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u(i,j+1) + \\ &+ \begin{bmatrix} 3 \\ 0 \end{bmatrix} u(i,j) + \begin{bmatrix} 2 \\ 2 \end{bmatrix} u(i-1,j) + \begin{bmatrix} 3 \\ 3 \end{bmatrix} u(i-1,j-1) \end{aligned} \tag{30a}$$

$$y(i,j) = [0 \quad 1] x(i,j) + 2u(i,j) \tag{30b}$$

Case 2. First we shall considered the proposed method for $N = 6$ and $M = 4$. We choose $n = 2$ and $n_1 = 3$, $n_2 = 2$ and we assume that

$$\begin{aligned} a_{6-k,l} &= a_{k,4-l} = 0, \quad 0 \leq k \leq 2 \\ b_{6-k,l} &= b_{k,4-l} = 0, \quad 0 \leq l \leq 1 \end{aligned} \tag{31}$$

Under this assumption we can modify the state variables diagram to the equivalent one shown in Fig. 2.

Using the state variables diagram we may write the following equations

$$\begin{aligned}x_1(i+1,j+1) &= a_{00}x_2(i-2,j-1)+a_{01}x_2(i-2,j)+a_{02}x_2(i-2,j+1)+a_{10}x_2(i-1,j-1)+a_{11}x_2(i-1,j)+a_{12}x_2(i-1,j+1)+\\ &+a_{20}x_2(i,j-1)+a_{21}x_2(i,j)+a_{22}x_2(i,j+1)+a_{30}x_2(i+1,j-1)+a_{31}x_2(i+1,j)+b_{00}u(i-2,j-1)+b_{01}u(i-2,j)+b_{02}u(i-2,j+1)+\\ &+b_{10}u(i-1,j-1)+b_{11}u(i-1,j)+b_{12}u(i-1,j+1)+b_{20}u(i,j-1)+b_{21}u(i,j)+b_{22}u(i,j+1)+b_{30}u(i+1,j-1)+b_{31}u(i+1,j)\end{aligned} \tag{32}$$

$$\begin{aligned}x_2(i+1,j+1) &= a_{63}x_2(i+1,j)+a_{62}x_2(i+1,j-1)+a_{54}x_2(i,j+1)+a_{53}x_2(i,j)+a_{52}x_2(i,j-1)+a_{44}x_2(i-1,j+1)+a_{43}x_2(i-1,j)+\\ &+a_{42}x_2(i-1,j-1)+a_{34}x_2(i-2,j+1)+a_{33}x_2(i-2,j)+a_{32}x_2(i-2,j-1)+x_1(i-2,j-1)+b_{63}u(i+1,j)+b_{62}u(i+1,j-1)+\\ &+b_{54}u(i,j+1)+b_{53}u(i,j)+b_{52}u(i,j-1)+b_{44}u(i-1,j+1)+b_{43}u(i-1,j)+b_{42}u(i-1,j-1)+b_{34}u(i-2,j+1)+\\ &+b_{33}u(i-2,j)+b_{32}u(i-2,j-1)\end{aligned}$$

The equations (32) can be written in the form

$$\begin{aligned}x(i+1,j+1) &= \mathbf{A}_{-1,0}x(i+1,j)+\mathbf{A}_{-1,1}x(i+1,j-1)+\mathbf{A}_{0,-1}x(i,j+1)+\mathbf{A}_{0,0}x(i,j)+\mathbf{A}_{0,1}x(i,j-1)+\mathbf{A}_{1,-1}x(i-1,j+1)+\\ &+\mathbf{A}_{1,0}x(i-1,j)+\mathbf{A}_{1,1}x(i-1,j-1)+\mathbf{A}_{2,-1}x(i-2,j+1)+\mathbf{A}_{2,0}x(i-2,j)+\mathbf{A}_{2,1}x(i-2,j-1)+\mathbf{B}_{-1,0}u(i+1,j)+\\ &+\mathbf{B}_{-1,1}u(i+1,j-1)+\mathbf{B}_{0,-1}u(i,j+1)+\mathbf{B}_{0,0}u(i,j)+\mathbf{B}_{0,1}u(i,j-1)+\mathbf{B}_{1,-1}u(i-1,j+1)+\mathbf{B}_{1,0}u(i-1,j)+\\ &+\mathbf{B}_{1,1}u(i-1,j-1)+\mathbf{B}_{2,-1}u(i-2,j-1)+\mathbf{B}_{2,0}u(i-2,j)+\mathbf{B}_{2,1}u(i-2,j-1)\end{aligned} \tag{33}$$

where

$$\begin{aligned}&\mathbf{A}_{-1,0}=\begin{bmatrix}0 & a_{31}\\ 0 & a_{63}\end{bmatrix},\quad \mathbf{A}_{-1,1}=\begin{bmatrix}0 & a_{30}\\ 0 & a_{62}\end{bmatrix},\quad \mathbf{A}_{0,-1}=\begin{bmatrix}0 & a_{22}\\ 0 & a_{54}\end{bmatrix},\quad \mathbf{A}_{0,0}=\begin{bmatrix}0 & a_{21}\\ 0 & a_{53}\end{bmatrix},\\ &\mathbf{A}_{0,1}=\begin{bmatrix}0 & a_{20}\\ 0 & a_{52}\end{bmatrix},\quad \mathbf{A}_{1,-1}=\begin{bmatrix}0 & a_{12}\\ 0 & a_{44}\end{bmatrix},\quad \mathbf{A}_{1,0}=\begin{bmatrix}0 & a_{11}\\ 0 & a_{43}\end{bmatrix},\quad \mathbf{A}_{1,1}=\begin{bmatrix}0 & a_{10}\\ 0 & a_{42}\end{bmatrix},\\ &\mathbf{A}_{2,-1}=\begin{bmatrix}0 & a_{02}\\ 0 & a_{34}\end{bmatrix},\quad \mathbf{A}_{2,0}=\begin{bmatrix}0 & a_{01}\\ 0 & a_{33}\end{bmatrix},\quad \mathbf{A}_{2,1}=\begin{bmatrix}0 & a_{00}\\ 0 & a_{32}\end{bmatrix},\quad \mathbf{B}_{-1,0}=\begin{bmatrix}b_{31}\\ b_{63}\end{bmatrix},\\ &\mathbf{B}_{-1,1}=\begin{bmatrix}b_{30}\\ b_{62}\end{bmatrix},\quad \mathbf{B}_{0,-1}=\begin{bmatrix}b_{22}\\ b_{54}\end{bmatrix},\quad \mathbf{B}_{0,0}=\begin{bmatrix}b_{21}\\ b_{53}\end{bmatrix},\quad \mathbf{B}_{0,1}=\begin{bmatrix}b_{20}\\ b_{52}\end{bmatrix},\quad \mathbf{B}_{1,-1}=\begin{bmatrix}b_{12}\\ b_{44}\end{bmatrix},\\ &\mathbf{B}_{1,0}=\begin{bmatrix}b_{11}\\ b_{43}\end{bmatrix},\quad \mathbf{B}_{1,1}=\begin{bmatrix}b_{10}\\ b_{42}\end{bmatrix},\quad \mathbf{B}_{2,-1}=\begin{bmatrix}b_{02}\\ b_{34}\end{bmatrix},\quad \mathbf{B}_{2,0}=\begin{bmatrix}b_{01}\\ b_{33}\end{bmatrix},\quad \mathbf{B}_{2,1}=\begin{bmatrix}b_{00}\\ b_{32}\end{bmatrix}\end{aligned} \tag{34}$$

From the output equation

$$y(i,j)=x_2(i,j) \tag{35}$$

we have

$$\mathbf{C}=[0 \quad 1] \tag{36}$$

In general case the assumption (31) takes the form

$$\begin{aligned}a_{N-k,l}&=a_{k,M-l}=0\\ b_{N-k,l}&=b_{k,M-l}=0\end{aligned}\quad 0\le k\le n_1-1,\quad 0\le l\le n_2-1 \tag{37}$$

In a similar way in general case we obtain the matrices $\boldsymbol{A}_{kl}\in R^{nxm}$ of the forms

$$\mathbf{A}_{-1,0}=\begin{bmatrix}0 & \dots & 0 & a_{n_1,n_2-1}\\ 0 & \dots & 0 & a_{2n_1,2n_2-1}\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & 0 & a_{nn_1,nn_2-1}\end{bmatrix},\quad \mathbf{A}_{-1,1}=\begin{bmatrix}0 & \dots & 0 & a_{n_1,n_2-2}\\ 0 & \dots & 0 & a_{2n_1,2(n_2-1)}\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & 0 & a_{nn_1,nn_2-2}\end{bmatrix},\dots,\mathbf{A}_{-1,n_2-1}=\begin{bmatrix}0 & \dots & 0 & a_{n_1,0}\\ 0 & \dots & 0 & a_{2n_1,n_2}\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & 0 & a_{nn_1,(n-1)n_2}\end{bmatrix}, \tag{38}$$

$$\mathbf{A}_{0,-1}=\begin{bmatrix}0 & \dots & 0 & a_{n_1-1,n_2}\\ 0 & \dots & 0 & a_{2n_1-1,2n_2}\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & 0 & a_{nn_1-1,nn_2}\end{bmatrix},\quad \mathbf{A}_{0,0}=\begin{bmatrix}0 & \dots & 0 & a_{n_1-1,n_2-1}\\ 0 & \dots & 0 & a_{2n_1-1,2n_2-1}\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & 0 & a_{nn_1-1,nn_2-1}\end{bmatrix},\dots,\mathbf{A}_{0,n_2-1}=\begin{bmatrix}0 & \dots & 0 & a_{n_1-1,0}\\ 0 & \dots & 0 & a_{2n_1-1,n_2}\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & 0 & a_{nn_1-1,(n-1)n_2}\end{bmatrix},$$

....

$$\mathbf{A}_{n_1-1,-1} = \begin{bmatrix} 0 & \dots & 0 & a_{0,n_2} \\ 0 & \dots & 0 & a_{n_1,2n_2} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & a_{(n-1)n_1,nn_2} \end{bmatrix}, \quad \mathbf{A}_{n_1-1,0} = \begin{bmatrix} 0 & \dots & 0 & a_{0,n_2-1} \\ 0 & \dots & 0 & a_{n_1,2n_2-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & a_{(n-1)n_1,nn_2-1} \end{bmatrix}, \dots, \mathbf{A}_{-1,0} = \begin{bmatrix} 0 & \dots & 0 & a_{00} \\ 1 & \dots & 0 & a_{n_1,n_2} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & a_{(n-1)n_1,(n-1)n_2} \end{bmatrix}$$

and the matrices $\mathbf{B}_{kl} \in R^n$ of the form

(39)

$$\mathbf{B}_{-1,0} = \begin{bmatrix} b_{n_1,n_2-1} \\ b_{2n_1,2n_2-1} \\ \vdots \\ b_{nn_1,nn_2-1} \end{bmatrix}, \quad \mathbf{B}_{-1,1} = \begin{bmatrix} b_{n_1,n_2-2} \\ b_{2n_1,2(n_2-1)} \\ \vdots \\ b_{nn_1,nn_2-2} \end{bmatrix}, \dots, \mathbf{B}_{-1,n_2-1} = \begin{bmatrix} b_{n_1,0} \\ b_{2n_1,n_2} \\ \vdots \\ b_{nn_1,(n-1)n_2} \end{bmatrix},$$

$$\mathbf{B}_{0,-1} = \begin{bmatrix} b_{n_1-1,n_2} \\ b_{2n_1-1,2n_2} \\ \vdots \\ b_{nn_1-1,nn_2} \end{bmatrix}, \quad \mathbf{B}_{0,0} = \begin{bmatrix} b_{n_1-1,n_2-1} \\ b_{2n_1-1,2n_2-1} \\ \vdots \\ b_{nn_1-1,nn_2-1} \end{bmatrix}, \dots, \mathbf{B}_{0,n_2-1} = \begin{bmatrix} b_{n_1-1,0} \\ b_{2n_1-1,n_2} \\ \vdots \\ b_{nn_1-1,(n-1)n_2} \end{bmatrix},$$

....

$$\mathbf{B}_{n_1-1,-1} = \begin{bmatrix} b_{0,n_2} \\ b_{n_1,2n_2} \\ \vdots \\ b_{(n-1)n_1,nn_2} \end{bmatrix}, \quad \mathbf{B}_{n_1-1,0} = \begin{bmatrix} b_{0,n_2-1} \\ b_{n_1,2n_2-1} \\ \vdots \\ b_{(n-1)n_1,nn_2-1} \end{bmatrix}, \dots, \mathbf{B}_{n_1-1,n_2-1} = \begin{bmatrix} b_{00} \\ b_{n_1,n_2} \\ \vdots \\ b_{(n-1)n_1,(n-1)n_2} \end{bmatrix}$$

The matrix $\mathbf{C}$ has the form

$$\mathbf{C} = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \in \mathrm{R}_+^{1\times n} \qquad (40)$$

Theorem 3. If the conditions (37) are satisfied then there exists a positive realization of the form (38)-(40) if the coefficients a_{kl} and b_{kl} of the transfer function (14) are nonnegative.

The proof is similar to the one of Theorem 2.

If the conditions of Theorem 3 are met then a positive realization (3) of the transfer function (14) can be found by the use of the following procedure.

***Procedure 2*.**

Step 1: Knowing the degrees N and M of the denominator $d(z_1,z_2)$ of (11) choose n, n_1, n_2 such that $nn_1 = N$ and $nn_2 = M$.

Step 2: It is the same as in Procedure 1.

Step 3: Using (38)-(40) find the matrices $\mathbf{A}_{kl}$ and $\mathbf{B}_{kl}$ for $k = -1,0,\dots,n_1-1$; $l = -1,0,\dots,n_2-1$ and $\mathbf{C}$.

Remark 2. For different choice of the state variables we obtain different forms of the matrices $\mathbf{A}_{kl}$ and $\mathbf{B}_{kl}$

Example 2. Find a positive realization of the form (38)-(40) of the transfer function (25).

It is easy to verify that the transfer function (25) satisfies the conditions of Theorem 3.

Using Procedure 2 we obtain.

Step 1. In this case we choose $n = 2$, $n_1 = 2$ and $n_2 = 1$.

Step 2. The matrix D is given by (26) and the strictly proper transfer function has the form (27) ((28)).

Step 3. Using (38)-(40) we obtain (zero matrices are omitted)

$$\mathbf{A}_{-1,0}=\begin{bmatrix}0 & a_{20}\\ 0 & a_{41}\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\quad \mathbf{A}_{0,-1}=\begin{bmatrix}0 & a_{11}\\ 0 & a_{32}\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix},\quad \mathbf{A}_{1,-1}=\begin{bmatrix}0 & a_{01}\\ 0 & a_{22}\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},$$
$$\mathbf{A}_{1,0}=\begin{bmatrix}0 & a_{00}\\ 0 & a_{21}\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix},\quad \mathbf{B}_{-1,0}=\begin{bmatrix}b_{20}\\ b_{41}\end{bmatrix}=\begin{bmatrix}3\\ 0\end{bmatrix},\quad \mathbf{B}_{0,-1}=\begin{bmatrix}b_{11}\\ b_{32}\end{bmatrix}=\begin{bmatrix}3\\ 5\end{bmatrix},$$
$$\mathbf{B}_{1,-1}=\begin{bmatrix}b_{01}\\ b_{22}\end{bmatrix}=\begin{bmatrix}2\\ 0\end{bmatrix},\quad \mathbf{B}_{1,0}=\begin{bmatrix}b_{00}\\ b_{21}\end{bmatrix}=\begin{bmatrix}3\\ 2\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}0 & 1\end{bmatrix} \tag{41}$$

The desired positive realization of (25) is given by (26) and (41) and the 2D systems is described by the equations

$$x(i+1,j+1)=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x(i+1,j)+\begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}x(i,j+1)+\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x(i-1,j+1)+\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}x(i-1,j)+\begin{bmatrix}3\\ 0\end{bmatrix}u(i+1,j)+\begin{bmatrix}3\\ 5\end{bmatrix}u(i,j+1)+$$
$$+\begin{bmatrix}2\\ 0\end{bmatrix}u(i-1,j+1)+\begin{bmatrix}3\\ 2\end{bmatrix}u(i-1,j) \tag{42}$$

$$y(i,j)=\begin{bmatrix}0 & 1\end{bmatrix}x(i,j)+2u(i,j)$$

4. Concluding remarks

A new method for finding a positive realization of a given 2D transfer function has been proposed. Two cases of choices of the order of realizations have been considered. In the first case there always exists a positive realizations of a given transfer functions of the form (11) if its coefficients are nonnegative (Theorem 2). In the second case ($N = nn_1$, $M = nn_2$) there exists a positive realizations of (11) if additionally the conditions (37) are satisfied (Theorem 3). Procedures for finding positive 2D realizations have been proposed and illustrated by numerical examples.

It is worth to underline that for a given 2D transfer function may exists many positive realizations with different numbers of delays (Example 1 and 2) or may not exist any positive realizations, for example if the product of leading coefficients of the numerator and denominator is negative.

The presented method can be extended for multi-input multi-output 2D systems with delays in state and input vectors. Extension of this method for singular systems with delays is an open problem.

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