Some Extensions of the Controllability and Observability Tests to Linear Systems

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Abstract:
The new controllability tests of the pairs (f(A), MB), where f(A) is a function well-defined on the spectrum of the matrix A and M is a nonsingular matrix defined by (3.7). They are proposed and illustrated by numerical examples. In the dual versions the tests can be used to test the observability of the pairs (f(A), CM).

Keywords: *Controllability, observability, linear, system, test, example*

1. Introduction

The notions of controllability and observability introduced by Kalman [9] are the basic concepts of the modern control theory. Modified tests for the controllability and the observability have been proposed in [2]. The tests have been extended to other classes of linear continuous-time and discrete-time systems [3–5, 7, 8, 10–12, 14]. Global stability of discrete-time nonlinear systems with descriptor and fractional linear parts and scalar feedbacks has been analyzed in [6]. The stabilization of positive descriptor fractional discrete-time linear systems with two different fractional order by a decentralized controller has been investigated in [13].

In this paper the tests will be extended to the pair $(f(A), MB)$, where $f(A)$ is a function well-defined on the spectrum of the matrix A and M is a nonsingular matrix defined by (3.7). The paper is organized as follows: Section 2 offers basic definitions and theorems concerning controllability, stability and Frobenius canonical forms of the continuous-time linear systems that have been recalled. The controllability of linear systems with different single inputs have been considered in Section 3. The controllability of linear systems with functions of the state matrices has been investigated in Section 4. The general case has been analyzed in Section 5. Concluding remarks are given in Section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, I_n- the $n\times n$ identity matrix.

2. Preliminaries

Consider the continuous-time linear system

$$\dot{x} = Ax + Bu, \tag{2.1a}$$

$$y = Cx, \tag{2.1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 2.1. [2–4, 6, 10, 11] The continuous-time linear system (2.1) is called controllable if for given initial state $x(0) \in \Re^n$ and a given final state $x_f \in \Re^n$ there exists an input $u(t) \in \Re^m$ for $t \in [0, t_f]$ which steers the system from $x(0)$ to $x_f = x(t_f)$.

Theorem 2.1. [2–4, 6, 10, 11] The linear system (2.1) is controllable if and only if one of the following conditions is satisfied:

1. (Kalman condition)

$$\mathrm{rank}[B \quad AB \quad \dots \quad A^{n-1}B] = n \tag{2.2}$$

2. (Hautus condition)

$$\mathrm{rank}[I_n s - A \quad B] = n \text{ for } s \in C \text{ (the field of complex numbers).} \tag{2.3}$$

Definition 2.2. [2–4, 6, 10, 11] The linear system (2.1) is called observable if knowing its input $u(t) \in \Re^m$ and its output $y(t) \in \Re^p$ for $t \in [0 \quad t_f]$ it is possible find its unique initial condition $x(0) \in \Re^n$.

Theorem 2.2. [2–4, 6, 10, 11] The linear system (2.1) is observable if and only if one of the following conditions is satisfied:

1. (Kalman condition)

$$\mathrm{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \tag{2.4}$$

2. (Hautus condition)

$$\mathrm{rank}\begin{bmatrix} I_n s - A \\ C \end{bmatrix} = n \text{ for } s \in C \text{ (the field of complex numbers).} \tag{2.5}$$

Definition 2.3. [2-4, 6, 10, 11] The system (2.1) for $u(t) = 0$ is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \text{ for any } x(0) \in \Re_+^n. \tag{2.6}$$

Theorem 2.3. [2–4, 6, 10, 11] The linear system (2.1) for $u(t) = 0$ is asymptotically stable if and only if the eigenvalues of the matrix A satisfies the condition

$$\mathrm{Re}\lambda_{\mathrm{i}} < 0 \text{ for } i = 1, \ldots, n. \tag{2.7}$$

Definition 2.4. The matrix A has the Frobenius canonical form if it has one of the following forms:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix},$$

$$A_2 = A_1^T = \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & \ldots & 0 & -a_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \ldots & -a_1 & -a_0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & .. \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix},$$

$$A_4 = A_3^T = \begin{bmatrix} -a_{n-1} & 1 & 0 & \ldots & 0 \\ -a_{n-2} & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -a_1 & 0 & 0 & \ldots & 1 \\ -a_0 & 0 & 0 & \ldots & 0 \end{bmatrix}. \tag{2.8}$$

The inverse matrices of the matrices (2.8) have also the Frobenius canonical forms [6].

It is well-known that [4, 9, 10, 12] the controllable pair ($A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times 1}$) by the similarity transformation $\bar{x} = Px$ can be transform to its canonical form

$$A_1 = P_1^{-1}AP_1 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix},$$

$$B_1 = P_1^{-1}B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \det P_1 \neq 0 \tag{2.9a}$$

$$A_2 = P_2^{-1}AP_2 = \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & \ldots & 0 & -a_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{bmatrix},$$

$$B_2 = P_2^{-1}B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \det P_2 \neq 0 \tag{2.9b}$$

In a similar way we may obtain the remaining canonical forms of A_3 and A_4 given by (2.8).

The following elementary operations on real matrices will be used [3, 4, 7]:

1) Multiplication of any i-th row (column) by the number a. This operation will be denoted by $L[i \times a]$ for row operation and by $R[i \times a]$ for column operation.

2) Addition to any i-th row (column) of the j-th row (column) multiplied by any number b. This operation will be denoted by $L[i + j \times b]$ for row operation and by $R[i+j\times b]$ for column operation.

3) for interchange of rows and $R[i, j]$ for interchange of columns.

3. Controllability of a class of multi-inputs linear systems

Consider the matrix A in the canonical form given by (2.8) and the matrix $B \in \mathfrak{R}^{n\times 1}$ with only one nonzero entry equal 1.

Theorem 3.1. The above pair (A,B) is controllable if the matrix A is nonsingular ($\det A \neq 0$).

Proof. Let the matrix A has the form

$$A = \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & \ldots & 0 & -a_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{bmatrix} \tag{3.1a}$$

and the matrix B one of the following forms

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, B_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{3.1b}$$

The pair (A, B_1) is controllable since by Hautus condition (2.3)

$$\begin{aligned} &\mathrm{rank}\begin{bmatrix} I_n s - A & B_1 \end{bmatrix} \\ &= \mathrm{rank}\begin{bmatrix} s & 0 & \ldots & 0 & a_0 & 1 \\ -1 & s & \ldots & 0 & a_1 & 0 \\ 0 & -1 & \ldots & 0 & a_2 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & -1 & s + a_{n-1} & 0 \end{bmatrix} \\ &= n \text{ for } \forall s \in C. \end{aligned} \tag{3.2a}$$

The pair (A, B_2) is also controllable since

$$\begin{aligned} &\mathrm{rank}\begin{bmatrix} I_n s - A & B_2 \end{bmatrix} \\ &= \mathrm{rank}\begin{bmatrix} s & 0 & \ldots & 0 & a_0 & 0 \\ -1 & s & \ldots & 0 & a_1 & 1 \\ 0 & -1 & \ldots & 0 & a_2 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & -1 & s + a_{n-1} & 0 \end{bmatrix} \\ &= n \text{ for } \forall s \in C \end{aligned} \tag{3.2b}$$

and $a_0 \neq 0$.

Continuing this procedure after n steps we obtain

$$\operatorname{rank}\begin{bmatrix} I_n s - A & B_n \end{bmatrix} = \operatorname{rank}\begin{bmatrix} s & 0 & \dots & 0 & a_0 & 0 \\ -1 & s & \dots & 0 & a_1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & s & a_{n-2} & 0 \\ 0 & 0 & \dots & -1 & s + a_{n-1} & 1 \end{bmatrix} = n \text{ for } \forall s \in C \tag{3.2c}$$

Therefore, the pairs (A, B_i), $i = 1,2,\dots,n$ is controllable if $a_0 = \det A \neq 0$.

Example 3.1. Consider the pair (A,B), for

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \tag{3.3a}$$

and

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.3b}$$

Using the Kalman condition (2.2) for (3.3) we obtain

$$\operatorname{rank}\begin{bmatrix} B_1 & A_1 B_1 & A_1^2 B_1 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix} = 3, \tag{3.4a}$$

$$\operatorname{rank}\begin{bmatrix} B_2 & A_1 B_2 & A_1^2 B_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & -2 & 5 \end{bmatrix} = 3, \tag{3.4b}$$

$$\operatorname{rank}\begin{bmatrix} B_3 & A_1 B_3 & A_1^2 B_3 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix} = 3 \tag{3.4c}$$

and

$$\operatorname{rank}\begin{bmatrix} B_1 & A_2 B_1 & A_2^2 B_1 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3, \tag{3.5a}$$

$$\operatorname{rank}\begin{bmatrix} B_2 & A_2 B_2 & A_2^2 B_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} = 3, \tag{3.5b}$$

$$\operatorname{rank}\begin{bmatrix} B_3 & A_2 B_3 & A_2^2 B_3 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & -1 & 3 \\ 0 & -2 & 5 \\ 1 & -3 & 7 \end{bmatrix} = 3. \tag{3.5c}$$

This confirm the Theorem 3.1.

Given the controllable pair

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix},$$

$$\bar{B} = P^{-1}B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \det A \neq 0, \quad \det P \neq 0 \tag{3.6}$$

Find a nonsingular matrix $M \in \Re^{n \times n}$ such that the pair $(\bar{A}, M\bar{B})$ is also controllable.

Let

$$M = P\bar{M}P^{-1} \tag{3.7}$$

then

$$P^{-1}MP = P^{-1}MPP^{-1}B = \bar{M}\,\bar{B} = \hat{B} \tag{3.8}$$

where

$$\bar{M} = P^{-1}MP, \quad \bar{B} = M^{-1}B \tag{3.9}$$

Let $\bar{M} \in \Re^{n \times n}$ be a monomial matrix (in each row and in each column only one element is nonzero and the remaining elements are zero). In this case the matrix $\bar{M}\,\bar{B} \in \Re^{n \times 1}$ has only one nonzero element and the remaining elements are zero and by Theorem 3.1 the pair $(\bar{A}, \hat{B})$ is controllable.

Therefore, the following theorem has been proved:

Theorem 3.2. If the pair (A,B) is controllable and $\det A \neq 0$ then the pair $(\bar{A}, \hat{B})$ is also controllable.

To find the matrix M the following procedure can be used.

Procedure 3.1.

Step 1. For the controllable pair (A,B), $\det A \neq 0$, find the matrix P and the pair $(\bar{A}, \bar{B})$ defined by (3.6).
Step 2. Choose the monomial matrix $\bar{M}$.
Step 3. Compute the desired matrix (3.7).

Example 3.2. For the given controllable pair

$$A = \begin{bmatrix} -0.5 & -1 & -0.75 \\ -0.5 & 1 & 0.75 \\ 1 & -6 & -2.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \tag{3.10}$$

find the matrix M such that the pair (A,MB) is also controllable.

Using Procedure 3.1 we obtain

Step 1. In this case the matrix P has the form

$$P = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{3.11}$$

and

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -2 \end{bmatrix}, \quad \bar{B} = P^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.12}$$

Step 2. We choose the monomial matrix $\bar{M}$ in the form

$$\bar{M} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.13}$$

Step 3. The desired matrix M has the form

$$M = P\bar{M}P^{-1} = \begin{bmatrix} 0 & -1 & -0.25 \\ -0.25 & 0 & 0 \\ 0.5 & -3 & -1.25 \end{bmatrix} \tag{3.14}$$

It is easy to check that the pair (A, MB) for A and B given by (3.10) and M by (3.14) is also controllable.

Remark 3.1.
The above considerations for single-input systems can be easily extended to multi-input case in the following way. Let $B \in \Re^{n\times m}, \quad m > 1$ and

$$Bk = b \in \Re^n, \quad k \in \Re^m. \tag{3.15}$$

The vector k is chosen so that the equivalent single input system with input matrix b is controllable and satisfies the condition

$$\det\begin{bmatrix} Bk & ABk & \dots & A^{n-1}Bk \end{bmatrix} \neq 0. \tag{3.16}$$

4. Controllability of systems with functions of state matrices

Consider the controllable linear system (2.1a) and the minimal characteristic polynomial of the matrix A in the form

$$\Psi(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}\dots(\lambda - \lambda_r)^{m_r} \tag{4.1}$$

where $\lambda_1, \dots, \lambda_r$ are the eigenvalues of the matrix A and $\sum_{i=1}^{r} m_i = m \le n$. It is assumed that the function $f(\lambda)$ is well defined on the spectrum $\sigma_A = \{\lambda_1, \dots, \lambda_r\}$ of the matrix A, i.e.

$$f(\lambda), \quad f^{(1)}(\lambda_k) = \left.\frac{df(\lambda)}{d\lambda}\right|_{\lambda=\lambda_k}, \quad \dots, \quad f^{(m_k-1)}(\lambda_k) = \left.\frac{d^{m_k-1}f(\lambda)}{d\lambda^{m_k-1}}\right|_{\lambda=\lambda_k}, \quad k = 1, \dots, r \tag{4.2}$$

are finite [1].

In this case the matrix $f(A)$ is well-defined and it is given by the Lagrange-Sylvester formula [1]

$$f(A) = \sum_{i=1}^{r} Z_{i1}f(\lambda_i) + Z_{i2}f^{(1)}(\lambda_i) + \dots + Z_{im_i}f^{(m_i-1)}(\lambda_i) \tag{4.3a}$$

where

$$Z_{ij} = \sum_{k=j-1}^{m_i-1} \frac{\Psi_i(A)(A - \lambda_i I_n)^k}{(k-j+1)!(j-1)!} \frac{d^{k-j+1}}{d\lambda^{k-j+1}} \left.\left[\frac{1}{\Psi_i(\lambda)}\right]\right|_{\lambda=\lambda_i} \tag{4.3b}$$

and

$$\Psi_i(\lambda) = \frac{\Psi(\lambda)}{(\lambda - \lambda_i)^{m_i}}, \quad i = 1, \dots, r. \tag{4.3c}$$

In particular case when the eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix A are distinct $(\lambda_i \neq \lambda_j, \quad i \neq j)$

$$\Psi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n) \tag{4.4}$$

then the formula (4.3a) has the form

$$f(A) = \sum_{k=1}^{n} Z_k f(\lambda_k) \tag{4.5a}$$

where

$$Z_k = \prod_{\substack{i=1 \\ i\neq k}}^{n} \frac{A - \lambda_i I_n}{\lambda_k - \lambda_i}, \qquad k = 1, \dots, n \tag{4.5b}$$

It is well-known [1] that the matrices Z_{ij} satisfy the equalities

$$\sum_{i=1}^{r} Z_{i1} = I_n \tag{4.6a}$$

$$Z_{ij}Z_{kj} = 0 \text{ for } i \neq k \tag{4.6b}$$

$$Z_{i1}Z_{kj} = Z_{ij} \text{ for } i = 1, \dots, r; \; j = 1, \dots, m \tag{4.6c}$$

$$Z_{i1}^k = Z_{i1} \text{ for } k = 1, 2, \dots; \; i = 1, \dots, r \tag{4.6d}$$

$$Z_{ij} = \frac{1}{(j-1)!}(A - I_n\lambda_i)^{j-1}Z_{i1} \text{ for } i = 1, \dots r; \; j = 1, \dots, m \tag{4.6e}$$

In particular case the matrices (4.5b) satisfy the equalities

$$\sum_{k=1}^{n} Z_k = I_n \tag{4.7a}$$

$$Z_iZ_j = 0 \text{ for } i \neq j, \; i, j = 1, \dots, n \tag{4.7b}$$

$$Z_i^k = Z_i \text{ for, } k = 1, 2, \dots \; i = 1, \dots, n \tag{4.7c}$$

To simplify the notation, we shall consider the details for the case when the matrix A has only distinct eigenvalues $\lambda_i \neq \lambda_j, \quad i \neq j$.

Theorem 4.1. The pair $(f(A), B)$ is controllable if, and only if, the pair (A, B) is controllable and the function $f(\lambda)$ is well defined on the spectrum of the matrix A.

Proof. Using (2.2) and (4.5) we obtain

$$\operatorname{rank}\begin{bmatrix} B & f(A)B & \dots & (f(A))^{n-1}B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B & \sum_{i=1}^{n} Z_i f(\lambda_i)B & \dots & \left(\sum_{i=1}^{n} Z_i f(\lambda_i)\right)^{n-1} B \end{bmatrix} \tag{4.8}$$

Taking into account that $Z_iZ_j = 0$ for $i \neq j$ and $Z_i^k = Z_i$ for $k = 1, 2, \dots$ $i = 1, \dots, n$ (relations (4.7)) we obtain

$$\operatorname{rank}\begin{bmatrix} B & f(A)B & \dots & (f(A))^{n-1}B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B & \sum_{i=1}^{n} Z_i f(\lambda_i)B & \dots & \sum_{i=1}^{n} Z_i (f(\lambda_i))^{n-1}B \end{bmatrix} \tag{4.9}$$

since

$$\left(\sum_{i=1}^{n} a_i Z_i\right)^k = \sum_{i=1}^{n} a_i^k Z_i \text{ for } k = 1, 2, \dots \tag{4.10}$$

After some algebraic manipulations we obtain

$$\begin{aligned}&\operatorname{rank}[B \quad f(A)B \quad \dots \quad (f(A))^{n-1}B] \\ &= \operatorname{rank}[B \quad AB \quad \dots \quad A^{n-1}B]\,H \\ &= \operatorname{rank}[B \quad AB \quad \dots \quad A^{n-1}B] \qquad (4.11a)\end{aligned}$$

where the matrix

$$H = \begin{bmatrix} 1 & a_0 & \dots & a_0^{n-1} \\ 0 & a_1 & \dots & a_1^{n-1} \\ \dots & \dots & \dots & \dots \\ 0 & a_{n-1} & \dots & a_{n-1}^{n-1} \end{bmatrix} \qquad (4.11b)$$

is nonsingular for $s_i \neq s_j, \quad i \neq j, \quad i,j = 1,\dots,n.$
This completes the proof.

Remark 4.1. In general case the proof of Theorem 4.1 is similar. In this case the relations (4.6) should be used instead of (4.7).

The controllability of the pair ($f(A)$, B) can be checked by the use of the following procedure.

Procedure 4.1.

Step 1. Compute the eigenvalues $\lambda_1,\dots,\lambda_n$ of the matrix A.
Step 2. Check if the function $f(\lambda)$ satisfies the conditions (4.2).
Step 3. Check the controllability of the pair ($f(A)$,B).

Example 4.1. Consider the controllable pair

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \qquad (4.12)$$

Using Procedure 4.1 we obtain the following.
Step 1. The characteristic polynomial of the matrix A given by (4.12) has the form

$$\det[I_3 s - A] = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{vmatrix} = s^3 + 6s^2 + 11s + 6 \qquad (4.13)$$

and the eigenvalues of the matrix are $s_1 = -1, \quad s_2 = -2, \quad s_3 = -3$.
Step 2. In this case the conditions (4.2) are satisfied and using (4.5b) we obtain

$$\begin{aligned} Z_1 &= \frac{(A - I_3 s_2)(A - I_3 s_3)}{(s_1 - s_2)(s_1 - s_3)} = 3I_3 + \frac{5}{2}A + \frac{1}{2}A^2, \\ Z_2 &= \frac{(A - I_3 s_1)(A - I_3 s_3)}{(s_2 - s_1)(s_2 - s_3)} = -3I_3 - 4A - A^2, \\ Z_3 &= \frac{(A - I_3 s_1)(A - I_3 s_2)}{(s_3 - s_2)(s_3 - s_1)} = I_3 + \frac{3}{2}A + \frac{1}{2}A^2 \quad (4.14)\end{aligned}$$

The matrices (4.14) satisfy the relations (4.7) and

$$e^A = a_0 I_3 + a_1 A + a_2 A^2 \qquad (4.15a)$$

where

$$a_0 = 3(e^{-1} - e^{-2}) + e^{-3}, \quad a_1 = \frac{5}{2}e^{-1} - 4e^{-2} + \frac{3}{2}e^{-3},$$
$$a_2 = \frac{1}{2}e^{-1} - e^{-2} + \frac{1}{2}e^{-3} \qquad (4.15b)$$

Step 3. Using (3.14) and (4.15a) we obtain

$$\begin{aligned}&\operatorname{rank}[B \quad e^A B \quad (e^A)^2 B] \\ &= \operatorname{rank}[B \quad (a_0 I_3 + a_1 A + a_2 A^2)B \quad (a_0 I_3 + a_1 A + a_2 A^2)^2 B] \\ &= \operatorname{rank}[B \quad a_0 B + a_1 AB + a_2 A^2 B \quad a_0^2 B + a_1^2 AB + a_2^2 A^2 B] \\ &= \operatorname{rank}[B \quad AB \quad A^2 B]\,H = \operatorname{rank}[B \quad AB \quad A^2 B] \quad (4.16a)\end{aligned}$$

where the matrix

$$H = \begin{bmatrix} 1 & a_0 & a_0^2 \\ 0 & a_1 & a_1^2 \\ 0 & a_2 & a_2^2 \end{bmatrix} \qquad (4.16b)$$

is nonsingular.

Therefore, the pair (e^A, B) is controllable.

5. General case and the extension to the observability of the systems

In this section the general case will be considered using the results of sections 3 and 4.

Theorem 5.1. If the function $f(\lambda)$ is well-defined on the spectrum of the matrix A, $\det A \neq 0$ and the pair (A,B) is controllable (A and B are defined by (3.6) and (3.8)) then the pair ($f(A)$,MB) is controllable for matrix M defined by (3.7).

Proof follows immediately from the proofs of Theorem 3.1 and 4.1.

The controllability of the pair ($f(A)$,MB) can be checked by the use of Procedures 3.1 and 4.1.

Example 5.1. Check the controllability of the pair (A^{-1}, MB) for the matrices A, B given by (3.10) and the matrix M by (3.14).

The pair (3.10) is controllable and the matrix M has been obtained from the matrix (3.13) by similarity transformation (3.14).

The eigenvalues of the matrix A are $s_1 = -1, \quad s_2 = -2, \quad s_3 = -3$ and the there exists the its inverse matrix

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -6 & -11 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} -1.833 & -1 & -0.167 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad (5.1)$$

Taking into account (3.14) and the matrix B we obtain

$$MB = \begin{bmatrix} 0 & -1 & -0.25 \\ -0.25 & 0 & 0 \\ -0.5 & -3 & -1.25 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.25 \\ -2 \end{bmatrix} \qquad (5.2)$$

The pair (A^{-1}, MB) is controllable since

$$\det[MB \quad A^{-1}MB \quad (A^{-1})^2 MB] = \begin{vmatrix} -0.5 & 1.5 & -2.208 \\ -0.25 & -0.5 & 1.5 \\ -2 & -0.25 & -0.5 \end{vmatrix} = -2.93 \qquad (5.3)$$

This simple example confirms the Theorem 5.1.

Using the well-known duality between the controllability and the observability of linear systems the above presented results for the controllability can be extended to the observability of the above classes of linear systems.

For example, the Theorem 5.1 can be extended to the observability case as follows.

Theorem 5.2. If the function $f(\lambda)$ is well-defined on the spectrum of the matrix A, $\det A \neq 0$ and the pair (A,C) is observable then the pair $(f(A),CM)$ is observable for the nonsingular matrix M defined by (3.7).

6. Concluding remarks

The controllability and observability tests have been extended to the pair $(f(A), MB)$, where $f(A)$ is a function well-defined on the spectrum of the matrix A and M is a nonsingular matrix defined by (3.7). The controllability tests of linear systems with different single inputs have been investigated (Theorem 3.1 and 3.2). The controllability tests of linear systems with functions of the state matrices has been established (Theorem 4.1). Procedures for checking the controllability tests have been given and illustrated by numerical examples. The tests can be easily extended to check the observability of the systems. Using the approach given in [5] the results of this paper can be extended to convex linear combination of the controllability (observability) pairs. Open problems are extensions of these results to discrete-time linear systems and to fractional linear systems.

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