Positivity and Stability of Descriptor Discrete-Time Linear Systems with Interval State Matrices

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Abstract:
The positivity and stability of descriptor discrete-time linear systems with interval state matrices are addressed. Necessary and sufficient conditions for the positivity of descriptor discrete-time linear systems are established. The stability of descriptor linear systems with interval state matrices is investigated.

Keywords: *Descriptor, Linear, Discrete-time, Interval system, Positivity, stability*

1. Introduction

A dynamical system is called positive if its state variables take nonnegative values for all nonnegative inputs and nonnegative initial conditions. Positive linear systems have been investigated in [1, 5, 10, 11], while positive nonlinear systems have been studied in [6, 7, 9, 17, 19].

Examples of positive systems include industrial processes involving chemical reactors, heat exchangers, and distillation columns, as well as storage systems, compartmental systems, and models for water and atmospheric pollution. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology, and medicine.

Positive linear systems with different fractional orders have been addressed in [3, 12, 14, 23]. Descriptor (singular) linear systems have been analyzed in [9, 15, 16], and the stability of a class of nonlinear fractional-order systems in [6, 13, 19, 26]. Application of Drazin inverse to the analysis of descriptor fractional discrete-time linear systems has been presented in [8], and stability of discrete-time switched systems with unstable subsystems in [24]. The robust stabilization of discrete-time positive switched systems with uncertainties has been addressed in [25]. Comparison of three methods of analysis of the descriptor fractional systems has been presented in [22]. Stability of linear fractional order systems with delays has been analyzed in [2], and simple conditions for practical stability of positive fractional systems have been proposed in [4]. The asymptotic stability of interval positive discrete-time linear systems has been investigated in [18].

In this paper, the positivity and stability of descriptor discrete-time linear systems with interval state matrices will be addressed.

The paper is organized as follows. In Section 2, some basic definitions and theorems related to descriptor discrete-time linear systems are reviewed. In Section 3, the positivity of descriptor discrete-time linear systems is investigated. The stability of positive descriptor linear discrete-time systems is analyzed in Section 4, and the stability of positive descriptor linear systems with interval state matrices is analyzed in Section 5. Concluding remarks are given in section 6.

The following notations will be used: R- the set of real numbers, R^{nxm} - the set of nxm real matrices, R_{+}^{nxm} - the set of *nxm* real matrices with nonnegative entries and $R_{+}^{n} = R_{+}^{nx1}$, Z_{+} - the set of nonnegative integers, I_{n-} - the *nxn* identity matrix, for A=[a_{ij}]∈ R^{nxm} and B=[b_{ij}] ∈ R^{nxm} and inequality A =B means a_{ij} = b_{ij} for i,j=1.2,...n.

2. Preliminaries

Consider the autonomous descriptor discrete-time linear system

$$Ex_{i+1} = Ax_i, i \in Z_+ = \{0, 1, ...\}, \quad (1)$$

where $x_i \in \Re^n$ is the state vector and $E, A \in \Re^{n\times n}$.

It is assumed that

$$\det[Ez - A] \neq 0 \text{ for some } z \in \mathbf{C} \text{ (the field of complex numbers)} \quad (2)$$

In this case, the system (1) has a unique solution for admissible initial conditions $x_0 \in \Re_+^n$.

It is well-known [20] that if (2) holds, then there exists a pair of nonsingular matrices $P, Q \in \Re^{n\times n}$ such that

$$P[Ez - A]Q = \begin{bmatrix} I_{n_1}z - A_1 & 0 \\ 0 & Nz - I_{n_2} \end{bmatrix},$$
$$A_1 \in \Re^{n_1\times n_1}, N \in \Re^{n_2\times n_2}, \quad (3)$$

where $n_1 = \deg\{\det[Ez - A]\}$ and N is the nilpotent matrix, i.e. $N^\mu = 0$, $N^{\mu-1} \neq 0$ (μ is the nilpotency index).

To simplify the considerations, it is assumed that the matrix N has only one block.

The nonsingular matrices P and Q can be found, for example, by the use of elementary row and column operations [20]:

1) Multiplication of any i-th row (column) by the number $c \neq 0$. This operation will be denoted by $L[i \times c]$ ($R[i \times c]$).

2) Addition to any i-th row (column) of the j-th row (column) multiplied by any number $c \neq 0$. This operation will be denoted by $L[i+j\times c]$ ($R[i+j\times c]$).
3) Interchange of any two rows (columns). This operation will be denoted by $L[i,j]$ ($R[i,j]$).

Definition 2.1. [5, 11] The autonomous discrete-time linear system

$$x_{i+1} = Ax_i, A \in \Re^{n\times n} \tag{4}$$

is called (internally) positive if $x_i \in \Re_+^n$, $i \in Z_+$ for all $x_0 \in \Re_+^n$.

Theorem 2.1. [5, 11] The system (4) is positive if and only if

$$A \in \Re_+^{n\times n}. \tag{5}$$

Definition 2.2. [5,11] The positive system (4) is called asymptotically stable (Schur) if

$$\lim_{i\to\infty} x_i = 0 \text{ for all } x_0 \in \Re_+^{n_1}. \tag{6}$$

Theorem 2.2. [18] The positive system (4) is asymptotically stable if and only if one of the equivalent conditions is satisfied:

1) All coefficients of the characteristic polynomial

$$\det[I_n(z+1)-A] = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 \tag{7}$$

are positive, i.e. $a_k > 0$ for $k = 0, 1, ..., n-1$.

2) There exists a strictly positive vector $\lambda = [\lambda_1 \quad \cdots \quad \lambda_n]^T, \lambda_k > 0, k = 1, \dots, n$ such that

$$A\lambda < \lambda. \tag{8}$$

3. Positive Descriptor Linear Systems

In this section, the necessary and sufficient conditions for the positivity of the descriptor linear discrete-time systems will be established.

Definition 3.1. The descriptor system (1) is called (internally) positive if $x_i \in \Re_+^n$, $i \in Z_+$ for all admissible nonnegative initial conditions $x_0 \in \Re_+^n$.

Theorem 3.1. The descriptor system (1) is positive if and only if the matrix E has only linearly independent columns, and the matrix $A_1 \in \Re_+^{n_1\times n_1}$.

Proof. Using the column permutation (the matrix Q) we choose n_1 linearly independent columns of the matrix E as its first columns. Next, using elementary row operations (the matrix P), we transform the matrix E to the form $\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}$ and the matrix A to the form $\begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}$. From (2), it follows that the system (1) has been decomposed into the following two independent subsystems

$$x_{1,i+1} = A_1 x_{1,i}, x_{1,i} \in \Re^{n_1}, i \in Z_+ \tag{9}$$

and

$$Nx_{2,i} = x_{2,i}, x_{2,i} \in \Re^{n_2}, i \in Z_+ \tag{10}$$

where

$$Q^{-1}x_i = \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}, i \in Z_+ \tag{11}$$

and Q and Q^{-1} are permutation matrices.

Note that the solution $x_{1,i} = A_1^i x_{10}$, $i \in Z_+$ of (9) is nonnegative if and only if $A_1 \in \Re_+^{n_1\times n_1}$ and the solution $x_{2,i}$ of (10) is zero for $i = 1, 2, ...$

Example 3.1. Consider the descriptor system (1) with the matrices

$$E = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & -\frac{2}{3} & 0 & -1 \\ 0 & \frac{2}{3} & 1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix}. \tag{12}$$

The condition (2) is satisfied since

$$\det[Ez-A] = \begin{vmatrix} 0 & -1 & 0 & 2z-1 \\ -1 & z+\frac{2}{3} & 0 & -2z+1 \\ z & -2z-\frac{2}{3} & -1 & 0 \\ -1 & 1 & 0 & -2z+1 \end{vmatrix} = -2z^2 + \frac{5}{3}z - \frac{1}{3} \tag{13}$$

and $n_1 = 2$. In this case $rankE = 3$ and $\mu = rankE - n_1 + 1 = 2$. Performing on the matrix

$$Ez - A = \begin{bmatrix} 0 & -1 & 0 & 2z-1 \\ -1 & z+\frac{2}{3} & 0 & -2z+1 \\ z & -2z-\frac{2}{3} & -1 & 0 \\ -1 & 1 & 0 & -2z+1 \end{bmatrix} \tag{14}$$

the following column shows elementary operations $R[4\times\frac{1}{2}]$, $R[4,1]$ and the row operations $L[2+4\times(-1)]$, $L[4+1\times 1]$, $L[3+2\times 2]$ we obtain

$$A_1 = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{3} \end{bmatrix}, N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{15}$$

In this case, the matrices Q and P have the forms

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 1 & -2 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \tag{16}$$

By Theorem 3.1, the descriptor system (1) with (12) is positive since $A_1 \in \Re_+^{2\times 2}$ and the matrix Q is monomial.

4. Stability of Positive Descriptor Linear Systems

Consider the descriptor system (1) satisfying the condition (2).

Lemma 4.1. The characteristic polynomials of the system (1) and of the matrix $A_1 \in \Re^{n_1\times n_1}$ are related by

$$\det[I_{n_1}z - A_1] = c\det[Ez-A], \tag{17}$$

where $c = (-1)^{n_2}\det P \det Q$.

Proof. From (2) we have

$$det[I_{n_1}z - A_1] = (-1)^{n_2} \det \begin{bmatrix} I_{n_1}z - A_1 & 0 \\ 0 & Nz - I_{n_2} \end{bmatrix} = (-1)^{n_2} \det P \det[Ez - A] \det Q = c \det[Ez - A]. \quad (18)$$

Theorem 4.1. The positive descriptor system (1) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficients of the characteristic polynomial

$$\det[I_{n_1}(z+1) - A_1] = z^{n_1} + a_{n_1-1}z^{n_1-1} + \dots + a_1 z + a_0 \quad (19)$$

are positive, i.e. $a_k > 0$ for $k = 0, 1, ..., n_1 - 1$.

2) All coefficients of the characteristic equation of the matrix $Ez - A$

$$\det[E(z+1) - A] = \bar{a}_{n_1}z^{n_1} + \bar{a}_{n_1-1}z^{n_1-1} + \dots + \bar{a}_1 z + \bar{a}_0 = 0 \quad (20)$$

are positive.

3) There exists a strictly positive vector $\lambda = [\lambda_1 \ \cdots \ \lambda_{n_1}]^T$, $\lambda_k > 0$, $k = 1, ..., n_1$ such that

$$A_1\lambda < \lambda. \quad (21)$$

4) There exists a strictly positive vector $\bar{\lambda} = [\bar{\lambda}_1 \ \cdots \ \bar{\lambda}_{n_1}]^T$, $\bar{\lambda}_k > 0$, $k = 1, ..., n_1$ such that

$$\bar{P}\tilde{A}\bar{\lambda} < \bar{\lambda}, \quad (22a)$$

where

$$\bar{P} = \bar{Q}_{n_1}P_{n_1}, \quad (22b)$$

$\bar{Q}_{n_1} \in \Re_+^{n_1\times n_1}$ consists of n_1 nonzero rows of $Q_{n_1} \in \Re_+^{n\times n_1}$ which is built of first n_1 columns of the matrix Q defined by (2),

$P_{n_1} \in \Re^{n_1\times n}$ consists of n_1 rows of the matrix P defined by (2),

$\tilde{A} \in \Re^{n\times n_1}$ consists of n_1 columns of $A \in \Re^{n\times n}$ corresponding to the nonzero rows of Q_{n_1}.

Proof. Proof of condition 1) follows immediately from condition 1) of Theorem 2.2. By Lemma 4.1 $\det[I_{n_1}(z+1) - A_1] = 0$ if and only if $\det[E(z+1) - A] = 0$. Therefore, the positive descriptor system (1) is asymptotically stable if and only if all coefficients of (20) are positive.

From (2), we have

$$A_1 = P_{n_1}AQ_{n_1} \quad (23)$$

and using (8), we obtain

$$A_1\lambda = P_{n_1}AQ_{n_1}\lambda < \lambda \quad (24)$$

for some strictly positive vector $\lambda \in \Re_+^{n_1}$. Premultiplying (24) by $\bar{Q}_{n_1}$ and taking into account $\bar{Q}_{n_1}\lambda = \bar{\lambda}$ and eliminating from A all columns corresponding to zero rows of Q_{n_1} we obtain (22).

Example 4.1. (Continuation of Example 3.1) Using Theorem 4.1, check the asymptotic stability of the positive descriptor system (1) with the matrices (12).

The matrix A_1 of the system is given by (15) and its characteristic polynomial

$$\det[I_2(z+1) - A_1] = \begin{bmatrix} z + \frac{1}{2} & -1 \\ 0 & z + \frac{1}{3} \end{bmatrix} = z^2 + \frac{7}{6}z + \frac{1}{3} \quad (25)$$

has positive coefficients. Therefore, by condition 1) of Theorem 4.1, the matrix A_1 is asymptotically stable.

The characteristic equation (20) of the matrices (12)

$$\det[E(z+1) - A] = \begin{vmatrix} 0 & -1 & 0 & 2z+1 \\ -1 & z + \frac{5}{3} & 0 & -2z-1 \\ z+1 & -2z - \frac{8}{3} & -1 & 0 \\ -1 & 1 & 0 & -2z-1 \end{vmatrix} = 2z^2 + \frac{7}{3}z + \frac{2}{3} = 0 \quad (26)$$

has positive coefficients and by condition 2) of Theorem 4.1, the positive system is asymptotically stable.

In this case we have

$$\bar{P} = \bar{Q}_{n_1}P_{n_1} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{A} = \begin{bmatrix} 1 & -1 \\ -\frac{2}{3} & 1 \\ \frac{2}{3} & 0 \\ -1 & 1 \end{bmatrix} \quad (27)$$

and

$$\bar{P}\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -\frac{2}{3} & 1 \\ \frac{2}{3} & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}. \quad (28)$$

Therefore, using (22a), (27), and (28), we obtain

$$\bar{P}\tilde{A}\bar{\lambda} = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (29)$$

and by condition (22), the positive system is asymptotically stable.

5. Stability of Positive Descriptor Linear Systems With Interval State Matrices

Consider the autonomous descriptor positive linear system

$$Ex_{i+1} = Ax_i, i \in Z_+ \quad (30)$$

where $x_i \in \Re^n$ is the state vector, $E \in \Re^{n\times n}$ is constant (exactly known) and $A \in \Re^{n\times n}$ is an interval matrix defined by

$$\underline{A} \le A \le \overline{A} \text{ or equivalently } A \in [\underline{A}, \overline{A}]. \quad (31)$$

It is assumed that

$$\det[Ez - \underline{A}] \neq 0 \text{ and } \det[Ez - \overline{A}] \neq 0 \quad (32)$$

and the matrix E has only linearly independent columns.

If these assumptions are satisfied, then there exist two pairs of nonsingular matrices (P_1, Q_1), (P_2, Q_2) such that

$$P_1[Es - \underline{A}]Q_1 = \begin{bmatrix} I_{\underline{n}_1} z - \underline{A}_1 & 0 \\ 0 & \underline{N}z - I_{\underline{n}_2} \end{bmatrix},$$
$$\underline{A}_1 \in \Re^{\underline{n}_1 \times \underline{n}_1}, \underline{N} \in \Re^{\underline{n}_2 \times \underline{n}_2},$$
$$\underline{n}_1 + \underline{n}_2 = n, \quad (33a)$$

and

$$P_2[Ez - \overline{A}]Q_2 = \begin{bmatrix} I_{\bar{n}_1} z - \overline{A}_1 & 0 \\ 0 & \overline{N}z - I_{\bar{n}_2} \end{bmatrix},$$
$$\overline{A}_1 \in \Re^{\bar{n}_1 \times \bar{n}_1}, \overline{N} \in \Re^{\bar{n}_2 \times \bar{n}_2},$$
$$\bar{n}_1 + \bar{n}_2 = n, \quad (33b)$$

where $\underline{n}_1 = \deg\{\det[Ez - \underline{A}]\}$ and $\bar{n}_1 = \deg\{\det[Ez - \overline{A}]\}$.

Theorem 5.1. If the assumptions are satisfied, then the interval descriptor system (30) with (31) is positive if and only if

$$\underline{A}_1 \in \Re_+^{\underline{n}_1 \times \underline{n}_1} \text{ and } \overline{A_1} \in \Re_+^{\bar{n}_1 \times \bar{n}_1}. \quad (34)$$

Proof. The proof is similar to the proof of Theorem 3.1.

Definition 5.1. The positive descriptor interval system (30) is called asymptotically stable (Schur) if the system is asymptotically stable for all matrices $E, A \in [\underline{A}, \overline{A}]$.

Theorem 5.2. If the matrices $\underline{A}$ and $\overline{A}$ of the positive system (30) is asymptotically stable, then its convex linear combination

$$A = (1-k)\underline{A} + k\overline{A} \text{ for } 0 \leq k \leq 1 \quad (35)$$

is also asymptotically stable.

Proof. By condition 2) of Theorem 2.2, if the positive systems are asymptotically stable, then there exists a strictly positive vector $\lambda \in \Re_+^n$ such that

$$\underline{A}\lambda < \lambda \text{ and } \overline{A}\lambda < \lambda. \quad (36)$$

Using (35) and (36), we obtain

$$\begin{aligned} A\lambda &= [(1-k)\underline{A} + k\overline{A}]\lambda \\ &= (1-k)\underline{A}\lambda + k\overline{A}\lambda < (1-k)\lambda + k\lambda \\ &= \lambda \text{ for } 0 \leq k \leq 1. \end{aligned} \quad (37)$$

Therefore, if the matrices $\underline{A}$ and $\overline{A}$ (36) hold, then the convex linear combination is also asymptotically stable.

Theorem 5.3. The positive descriptor system (30) with the matrix E with only linearly independent columns and interval matrix A is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \Re_+^n$ such that

$$\bar{P}\underline{A}\lambda < \lambda \text{ and } \bar{P}\bar{A}\lambda < \lambda, \quad (38)$$

where $\bar{P}$ is defined by (22b).

Proof. By assumption, the matrix E has only linearly independent columns and $\lambda = Q\lambda_q \in \Re_+^n$ is strictly positive for any $\lambda_q \in \Re_+^n$ with all positive components. By condition 2) of Theorem 2.2 and Theorem 5.2, the positive descriptor system with interval (31) is asymptotically stable if and only if the conditions (38) are satisfied.

Example 5.1. (Continuation of Example 4.1) Consider the positive descriptor system (30) with E given by (12) and the interval matrix A with

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & 0.4 \\ 0 & -0.7 & 0 & -0.4 \\ 1 & -0.6 & 1 & 0 \\ 0 & -1 & 0 & -0.4 \end{bmatrix},$$
$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 & 0.8 \\ 0 & -0.4 & 0 & -0.8 \\ 1 & -1.2 & 1 & 0 \\ 0 & -1 & 0 & -0.8 \end{bmatrix}. \quad (39)$$

We shall check the stability of the system using Theorem 5.3. The matrices Q and P have the same form (16) as in Examples 3.1 and 4.1. Therefore, the matrix $\bar{P}$ in (38) is the same as in Example 4.1, and it is given by (27). Taking into account that in this case

$$\underline{\tilde{A}} = \begin{bmatrix} 1 & 0.4 \\ -0.7 & -0.4 \\ -0.6 & 0 \\ -1 & -0.4 \end{bmatrix} \text{ and}$$
$$\overline{\tilde{A}} = \begin{bmatrix} 1 & 0.8 \\ -0.4 & -0.8 \\ -1.2 & 0 \\ -1 & -0.8 \end{bmatrix} \quad (40)$$

and using (38), we obtain

$$\bar{P}\underline{\tilde{A}}\bar{\lambda} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0.5 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.4 \\ -0.7 & -0.4 \\ -0.6 & 0 \\ -1 & -0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0.3 & 0 \\ 0.5 & 0.2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (41a)$$

and

$$\bar{P}\overline{\tilde{A}}\bar{\lambda} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0.5 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.8 \\ -0.4 & -0.8 \\ -1.2 & 0 \\ -1 & -0.8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0.6 & 0 \\ 0.5 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (41b)$$

Therefore, by Theorem 5.3, the positive descriptor system is asymptotically stable.

6. Concluding Remarks

The positivity and asymptotic stability of descriptor linear discrete-time systems have been addressed. Necessary and sufficient conditions for the positivity (Theorem 3.1) of the descriptor linear discrete-time systems and for the asymptotic stability (Theorem 4.1) of positive descriptor systems have been established. It has been shown that the descriptor linear systems are positive if and only if the conditions (34) are satisfied (Theorem 5.1). Necessary and sufficient conditions for the asymptotic stability of a positive descriptor linear system (30) with interval state matrices have also been established (Theorem 5.3). Numerical examples of descriptor positive discrete-time linear systems have illustrated the considerations.

The considerations can be extended to continuous-time and discrete-time positive fractional linear systems.

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