A 2-tuple Linguistic Dynamic OWAWA Aggregation Operator and its application to multi-attribute decision-making

Submitted: 14th April 2024; accepted: 17th September 2024

Yeleny Zulueta-Véliz, Carlos Rafael Rodríguez Rodríguez, Aylin Estrada Velazco

DOI: 10.14313/jamris-2025-032

Abstract:
A linguistic dynamic decision-making problem reveals situations in which the decision data gathered in multiple periods is represented by means of linguistic values. To deal with linguistic variables in linguistic dynamic decision-making problems, the 2-tuple linguistic model stands out among computational models because of its accuracy and interpretability. The selection of a suitable time-dependent 2-tuple linguistic aggregation operator is relevant due to its properties that can highly modify the computing cost as well as the results themselves and their accuracy and interpretability. This paper proposes a new 2-tuple linguistic dynamic hybrid weighted aggregation operator which is suitable to model different attitudes in decision-making by simultaneously weighting the given arguments as well as their ordered positions. The novel 2-tuple Linguistic Dynamic Ordered Weighted Averaging-Weighted Average (2TDOWAWA) operator weights not only the importance of a particular time period, but also the importance of non-dynamic evaluations in such a time period. Eventually, a 2-tuple Linguistic Dynamic Multiple Attribute Decision-Making approach based on the 2TDOWAWA Aggregation Operator is described. Finally a practical example is provided to illustrate the developed approach and to demonstrate its practicality and effectiveness.

Keywords: *linguistic decision-making, dynamic decision-making, 2-tuple linguistic model, aggregation operator*

1. Introduction

Granulation plays a key role in human cognition. For humans, it serves as a way of achieving data compression. This is one of the pivotal advantages accruing through the use of words in human, machine, and man-machine communication [1]. Zadeh emphasized two keynotes [2]: the concept of granulation is unique to fuzzy logic [3] and closely related to the concept of a rough set [4]. Granulation involves partitioning a set into granules and a granule may be interpreted as a restriction on the values that a variable can takes. In this sense, words in a natural language are, in large measure, labels of granules.

Since a linguistic variable is a variable whose values are words or, equivalently, granules, the concept of granulation is rooted in the concept of linguistic variables [5].

In general sense, by ”information granule”, one regards a collection of elements drawn together by their closeness (resemblance, proximity or functionality) articulated in terms of some useful spatial, temporal or functional relationships. When we decompose an uncertain decision-making problem into granules, temporal relationships are very important to focus on the most suitable level of detail.

As decision environments and contents become increasingly complex, the use of single-granule information alone fails to accurately describe dynamic, ambiguous and fragmentary cognitive information. Dynamic multi-attribute decision-making frameworks offer to decision-makers a way of dealing with uncertainty, since this kind of solution enables for an iterative and interactive process in which the decision information is usually collected from different period [6–9]. That is, the dynamic decision-making problem consists of selecting the best alternatives from a set of available ones but considering time granulation [10]. Dynamic multi-attribute decision-making approaches are implicitly granule-based because they generally model a dynamic problem as a collection of static decision-making problems that are solved first and then their results are aggregated using a dynamic weighted aggregation operator and its weighting vector.

The concept of the linguistic dynamic multi-attribute decision-making (LDMADM) problem reveals situations in which decision data gathered in multiple periods, which is represented by linguistic terms by means of linguistic variables.

With a granule being a collection of elements which are drawn together by equivalence, proximity, similarity or functionality, in the LDMADM process, uncertainty is managed in these two granule-based dimensions: (a) linguistic variables and (b) time periods. Let's get into the twofold complexity in brief.

To deal with (a) linguistic variables in LDMADM, the 2-tuple linguistic representation model [11] provides a powerful approach because it can express any counting of information in the discourse universe; meanwhile, it improves the interpretability and effectiveness of the decision-making results by avoiding losing information in computations. Studies of 2-tuple linguistic representation model not only have a strong theoretical research value, but also have wide application prospects in practice, specifically in decision-making and decision analysis.

To deal with (b) time periods in LDMADM, the resolution process has been structured in [12]. The selection of a suitable time-dependent linguistic aggregation operator, and its weighting vector if necessary, is a key element due to the properties that can highly modify the computing cost as well as results themselves and their accuracy and interpretability. The aggregation is a multi-step process: first, a collective assessment is calculated for each alternative for each period, i.e., each static problem is solved; second, a dynamic collective assessment for each alternative is calculated using values obtained previously, i.e., the general dynamic problem is solved and an overall result is obtained.

This dynamic aggregation is generally carried out using time-dependent aggregation operators considering the diverse influence of time periods in results by means of weighting vectors. Based on the above reviews, we face the need of proper 2-tuple linguistic aggregation operators for such a time-dependent aggregation process.

What kind of 2-tuple linguistic time-dependent aggregation operators are available in the literature? As far as we knows, the 2-tuple linguistic Dynamic Weighted Averaging (2TDWA) [14], the 2-tuple linguistic Dynamic Averaging (2TDA) [14], the 2-tuple linguistic Dynamic Weighted Geometric (2TDWG) [15] and the 2-tuple linguistic Dynamic Geometric (2TDG) [15] aggregation operators only weight the 2-tuple linguistic arguments themselves. That is, they weight each time period in relation to their reliability but they can not synthetically consider the importance of time periods and the importance of non-dynamic evaluations.

What kind of hybrid weighted aggregation operators are available in the literature? Numeric aggregation operators have been studied for a long time. Among the large number of aggregation operators and functions, the arithmetic mean (AM) and the weighted mean (WM) are the most popular ones. A related operator, the ordered weighted averaging (OWA) operator, was proposed by Yager in [16]. This operator is similar to the WM as both are a linear combination of the input data. The difference between the WM and the OWA operator is that the latter orders the data before applying the linear combination [17]. This ordering step causes the semantics (or meaning) of the weights to be radically different in the weighted mean and the OWA. In fact, the weights in the weighted mean measure the reliability of the sources and the weights in the OWA measure the importance of the values (with respect to their ordering). The need of combining both functions has been developed by different authors [18, 19] and three main classes of functions have been proposed for generalizing them: the weighted OWA (WOWA) operator [18], the hybrid weighted averaging(HWA) operator [19], and the ordered weighted averaging-weighted average (OWAWA) operator [20]. The main advantage of the last approach is that it unifies the OWA and the WA, taking into account the degree of importance that each concept has in the formulation.

Motivated by this gap, in this paper, we propose a new 2-tuple linguistic dynamic hybrid weighted aggregation operator which is useful to model different attitudes in decision-making by simultaneously weighting the given arguments as well as their ordered positions.

The remainder of this paper is structured as follows. Section 2 reviews basic concepts of the 2-tuple linguistic representation model. Section 3 introduces a new 2-tuple linguistic OWAWA aggregation operator which is integrated in the 2-tuple LDMADM approach described in Section 4. Section 5 gives an illustrative example, and Section 6 summarizes the key findings of this research.

2. Preliminaries

This section revises concepts and methods to be referred to in this paper, including the 2-tuple linguistic representation model and its computational model.

2.1. 2-tuple Linguistic Representation Model

In [11], Herrera and Martínez progressed the fuzzy linguistic decision-making field by representing the linguistic information with the name 2-tuple, constructed by a linguistic term and a numerical value, supporting the information of the symbolic translation.

The 2-tuple linguistic model [11] aimed to improve the accuracy and facilitate the processes of computing with words by treating the linguistic domain as continuous but keeping the linguistic basis (syntax and semantics). The 2-tuple fuzzy linguistic representation model consists of modelling the linguistic information by means of a pair of elements [21]:

- Let $S = \{s_0, ..., s_g\}$ be a linguistic term defined by the fuzzy linguistic approach whose semantics (provided by a fuzzy membership function) and syntax are also defined according to the fuzzy linguistic approach.
- α is a numerical value, *Symbolic Translation*, that indicates the translation of the fuzzy membership function which represents the closest term, $s_i \in \{s_0, ..., s_g\}$ if s_i does not match exactly the computed linguistic information. The value of α is then defined as

$$\alpha \in = \begin{cases} [-0.5, 0.5) & if \quad s_i \in \{s_1, s_2, ..., s_{g-1}\} \\ [0, 0.5) & if \quad s_i = s_0 \\ [-0.5, 0) & if \quad s_i = s_g \end{cases} \tag{1}$$

The linguistic information is then expressed by a pair of elements noted as (s_i, α). A symbolic computation on linguistic terms in S obtains a value $\beta \in [0, g]$ that will be transformed into a equivalent 2-tuple linguistic value, (s_i, α), by means of the Δ function defined as follows:

Definition 1 *[21] Let $S = \{s_0, ..., s_g\}$ the set of linguistic terms, the associated 2-tuple is $\tilde{S} = S \times [-0.5, 0.5)$ and the bijective function $\Delta : [0, g] \rightarrow \tilde{S}$ is given by:*

$$\Delta(\beta) = \begin{cases} s_i, & i = round(\beta) \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5) \end{cases} \quad (2)$$

with $round$ assigns to β the integer number $i \in \{0, 1, ..., g\}$ closest to β.

The Δ and Δ^{-1} transformation functions support conversions between numerical values and 2-tuple linguistic values without information loss. The 2-tuple linguistic model only guarantees accuracy when dealing with a uniformly and symmetrically distributed linguistic term set.

The recent two decades have witnessed the booming interest and growing development in research of 2-tuple linguistic time-independent aggregation operators. Functions Δ and Δ^{-1} greatly help the extension of conventional numerical operators to the 2-tuple linguistic domain. In what follows, two seminal 2-tuple linguistic time independent aggregation operators are revised.

Definition 2 *[11] Let $\tilde{L} = \{(s_1, \alpha_1), ..., (s_m, \alpha_m)\}$ be a set of 2-tuple linguistic values, and $W = (w_1, ..., w_m)$, $w_i \in [0, 1]$ be a weighting vector such that $\sum_{i=1}^{m} w_i = 1$, the weighted averaging aggregation operator associated with W is the function 2TWA: $\tilde{S}^m \to \tilde{S}$ defined as:*

$$2TWA(\tilde{L}) = \Delta\left(\sum_{i=1}^{m} w_i \Delta^{-1}(s_i, \alpha_i)\right) \quad (3)$$

Especially, if $W = \{\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m}\}$, the 2TWA operator reduces to the 2-tuple arithmetic mean (2TAM) operator:

$$2TAM(\tilde{L}) = \Delta\left(\frac{1}{m}\sum_{i=1}^{m} \Delta^{-1}(s_i, \alpha_i)\right) \quad (4)$$

2.2. Discrete time 2-tuple linguistic variable

The significant characteristic of the 2-tuple linguistic variable is that it involves the dimension of time, and this concept is pivotal in understanding 2-tuple LDMADM problems.

Definition 3 *[10] Let t be the variable of time, then $\tilde{L}(t)$ is called a discrete time 2-tuple linguistic variable where $\tilde{L}(t) = ((s_1, \alpha_1)(t_1), ..., (s_q, \alpha_q)(t_q))$ is a collection of q 2-tuple arguments collected from q different periods, $T = \{t_1, ..., t_q\}$.*

Operation laws and properties on the conventional 2-tuple linguistic value also hold for the discrete time 2-tuple linguistic variable because if omitting the parameter of the time (t_λ), the later can be mathematically taken as the former.

The concept of discrete time 2-tuple linguistic variable addresses the representation of changes of experts' assessments on given alternatives over an attribute but considers different time periods in the LDMADM process.

2.3. 2-tuple Dynamic Weighted Aggregation Operators

The 2-tuple linguistic aggregation operators are logically required to develop the dynamic aggregation phase in the LDMADM resolution process. Let's analyze some of the existing aggregation operators.

Definition 4 *[14] Let $\tilde{S} = \{(s_1, \alpha_1)(t_1), ..., (s_q, \alpha_q)(t_q)\}$ be a collection of q 2-tuple arguments collected from q different periods, $T = \{(t_\lambda)|\lambda \in (1, ..., q)\}$, whose weights are given by the weighting vector W, then the function $2TDWA : \tilde{S}^q \longrightarrow \tilde{S}$ defined as*

$$2TDWA(\tilde{S}) = \Delta\left(\sum_{\lambda=1}^{q} w(t_\lambda)\Delta^{-1}(s_i, \alpha_i)(t_\lambda)\right) \quad (5)$$

is called a 2-tuple Dynamic Weighted Averaging aggregation operator.

Especially, if $W^T = \{\frac{1}{q}, \frac{1}{q}, ..., \frac{1}{q}\}$, the 2TDWA operator reduces to the 2-tuple Dynamic Averaging (2TDA) aggregation operator:

$$2TDA(\tilde{S}) = \Delta\left(\frac{1}{q}\sum_{\lambda=1}^{q} \Delta^{-1}(s_i, \alpha_i)(t_\lambda)\right) \quad (6)$$

Definition 5 *[22] Let $\tilde{L} = \{(s_1, \alpha_1)(t_1), ..., (s_q, \alpha_q)(t_q)\}$ be a collection of q 2-tuple arguments collected from q different periods, $T = \{(t_\lambda)|\lambda \in (1, ..., q)\}$ and a weighting vector W, then the function $2TDOWA : \tilde{S}^q \longrightarrow \tilde{S}$ defined as*

$$2TDOWA(\tilde{L}) = \Delta\left(\sum_{j=1}^{q} w_i \Delta^{-1}(s_j, \alpha_j)(t_\lambda)\right) \quad (7)$$

is called a 2-tuple Dynamic Ordered Weighted Averaging aggregation operator, where $(s_j, \alpha_j)(t_\lambda)$ is the j-th largest of the $(s_i, \alpha_i)(t_\lambda)$ values.

On the one hand, in the 2TDWA the *i*-th 2-tuple linguistic value is weighted according to the weight $w(t_\lambda)$. On the other hand, in 2TDOWA each $w(t_\lambda)$ is attached to the *i*-th value in decreasing order without considering from which information source the value comes. Notice that the OWA operator is commutative. That is, all information sources (or experts) have an equal contribution to the final solution.

3. The 2-tuple Linguistic Dynamic OWAWA Aggregation Operator

The behavior of weighted averaging operators allows us to weight each information source in relation to their reliability while ordered weighted operators allow to weight the values according to their ordering.

The 2TDWA [14, 15] operator only weights the 2-tuple arguments themselves, but ignores the importance of the ordered position of the arguments, while the 2TDOWA [22] operator only weights the ordered position of each given arguments, but ignores the importance of the arguments. To solve this drawback, a new 2-tuple aggregation operators will be defined for time dependent 2-tuple linguistic arguments, which weight all the given arguments and their ordered positions based on OWAWA operator [20].

In the rest of the paper, we will recall or introduce definitions of weighted aggregation operators. It is worth noting that these functions are defined by means of vectors with non-negative components whose sum is 1.

Definition 6 *A vector* $w \in \mathbb{R}^n$ *is a weighting vector if* $w \in [0,1]^n$ *and* $\sum_{i=1}^{n} w_i = 1$

In the following, W is a weighting vector defined on the ordered set, while V is a weighting vector defined on the discrete time 2-tuple linguistic set, which is defined on the time period set.

Definition 7 *Let* $\tilde{L}(t) = \{(s_i, \alpha_i)(t_\lambda) | i \in (1, ..., q), \lambda \in (1, ..., q)\}$ *be a collection of* q *discrete time 2-tuple linguistic arguments on* $\tilde{S}$*, whose weights are given by the weighting vector* $V = \{v_\lambda | \lambda \in (1, ..., q)\}$*; and* $\ell \in [0,1]$*. Then the function* $2TDOWAWA : \tilde{S}^q \longrightarrow \tilde{S}$ *defined as*

$$2TDOWAWA_{V,W}(\tilde{L}) = \Delta\left(\ell \sum_{\lambda=1}^{q} w_j \Delta^{-1}(s_j, \alpha_j)(t_\lambda) + (1-\ell) \sum_{\lambda=1}^{q} v_\lambda \Delta^{-1}(s_i, \alpha_i)(t_\lambda)\right) \quad (8)$$

is called a 2-tuple Linguistic Dynamic Ordered Weighted Averaging–Weighted Average, $2TDOWAWA_{V,W}$*, aggregation operator, where* (s_j, α_j) *is the* j*-th largest of the weighted 2-tuple linguistic values* $(s_i, \alpha_i)(t_\lambda)$ *values. The* ℓ *value represents the relevance of each weighted model in the aggregation.*

This formulation of the $2TDOWAWA_{V,W}$ operator separates the part that strictly affects the OWA operator and the part that affects the WA operator. In this way, we can see both models in the same formulation.

By modulating the ℓ coefficient in the $2TDOWAWA_{V,W}$ operator, we may construct diverse aggregation operators. If $\ell = 0$, then we get the 2TDWA while if $\ell = 1$, we get the 2TDOWA operator. It is also possible to obtain a wide range of particular $2TDOWAWA_{V,W}$ cases by giving different values and interpretations to the ℓ value. For instance, we may introduce the 2TDOWA with a low degree of importance, such as $\ell \in [0, 0.2]$, and analyse the effect in the outputs; or we may also introduce the 2TDOWA in such a way that it is more important than the WA, by considering higher degrees of importance such as $\ell \in [0.8, 1]$. The greater the ℓ value, the more important the 2TDOWA operator, and vice versa. Along the same line, we can introduce the WA to a problem formulated with the OWA.

Each family is just a particular case useful in some special situations according to the interests of the analysis.

4. A 2-tuple LDMADM Approach based on the *2TDOWAWA*$_{V,W}$ Aggregation Operator

In this section, a LDMADM Approach with the $2TDOWAWA_{V,W}$ is given to introduce how this operator can be used to support a decision.

Let $T = \{t_\lambda | \lambda \in (1, ..., q)\}$ denotes the discrete set of evaluation time periods and $V = \{v_\lambda | \lambda \in (1, ..., q)\}$ represents the time weighting set that satisfies $v_\lambda \in [0,1]$ with $\sum_{\lambda=1}^{q} v_\lambda = 1$ $A = \{a_i | i \in (1, ..., m)\}$, being a discrete set of alternatives which are evaluated by the set of experts $E = \{e_k | k \in (1, ..., p)\}$ whose weights are given by the weighting vector $U(t_\lambda) = (u_k^\lambda | k \in (1, ..., p), \lambda \in (1, ..., q))$.

Alternatives are evaluated according to a criteria set $C = \{c_j | j \in (1, ..., n)\}$ whose weights are given by the weighting vector $H(t_\lambda) = (h_j^\lambda | j \in (1, ..., n), \lambda \in (1, ..., q))$.

Note that we also suppose that weights h_j^λ and u_k^λ may change during the whole period T. The preference provided by expert $e_k \in E$ about alternative $a_i \in A$ according to criterion $c_j \in C$ is represented by a linguistic term $x_{ijk}(t_\lambda) \in S = \{s_0, ..., s_g\}$.

Then, the dynamic evaluation of alternatives is defined as a discrete time 2-tuple linguistic variable whose values can be considered as the non-dynamic evaluations generated during the period T, since we consider the temporal problem as a succession of q individual LDMADM problems.

The 2-tuple LDMADM approach based on the $2TDOWAWA_{V,W}$ Aggregation Operator is aimed to solve decision-making problems in which linguistic preferences are gathered in multiple periods and a final decision is made considering all the linguistic information provided. Then, all non-dynamic evaluations are considered to have been conducted in the past. In other words, the aim of the 2-tuple LDMADM approach is to give a global order of alternatives set A based on dynamic linguistic evaluations, with respect to the criteria set C and after being evaluated during the period T.

The $2TDOWAWA_{V,W}$ operator is applied to LDMADM problems based on 2-tuple linguistic information.

A stepwise description of the 2-tuple LDMADM approach is provided in the following.

Step 1: Generate 2-tuple linguistic values for each period. An original linguistic term can be directly written as $(s_i, 0)$ 2-tuple linguistic since s_i represents the linguistic label center of the information (s_i, α) and $\alpha = 0$ represents no difference from the original value β to the transformed value i. Then from gathered $x_{ijk}(t_\lambda) = s_{ijk}(t_\lambda) \in S$, we will obtain 2-tuple linguistic values $\tilde{x}_{ijk}(t_\lambda) = (s, 0)_{ijk}(t_\lambda) \in \tilde{S}$.

Step 2: Calculate the collective value for each criterion for each period, using a classical time independent 2-tuple linguistic aggregation operator Υ_U and the weighting vector U.

$$\hat{X}(t_\lambda) = \left(\hat{x}_{ij}(t_\lambda)\right)_{m\times n} = \Upsilon_U\left(\tilde{x}_{ijk}(t_\lambda)\right) \quad (9)$$

Step 3: Calculate the non-dynamic evaluation for each alternative for each period, using a classical time independent 2-tuple linguistic aggregation operator Ψ_H and the weighting vector H. Results from this step can be seen as solutions for each individual or static LDMADM problem.

$$\bar{X}(t_\lambda) = (\bar{x}_i(t_\lambda))_m = \Psi_H\left(\hat{x}_{ij}(t_\lambda)\right) \quad (10)$$

Step 4: Calculate the dynamic evaluation for each alternative, if no other period will be considered in the LDMADM problem, using the $2TDOWAWA_{V,W}$.

$$\check{X}(t_\lambda) = (\check{x}_i(t_\lambda))_m = 2TDOWAWA_{V,W}\left(\bar{x}_i(t_\lambda)\right) \quad (11)$$

Step 5: Order dynamic evaluation $\check{x}_i(t_\lambda)$ values to obtain the best alternative(s). The $\check{x}_i(t_\lambda)$ value is such that the higher this score, the better the alternative a_i is ranked.

The flowchart of the proposed approach is presented in Figure 1.

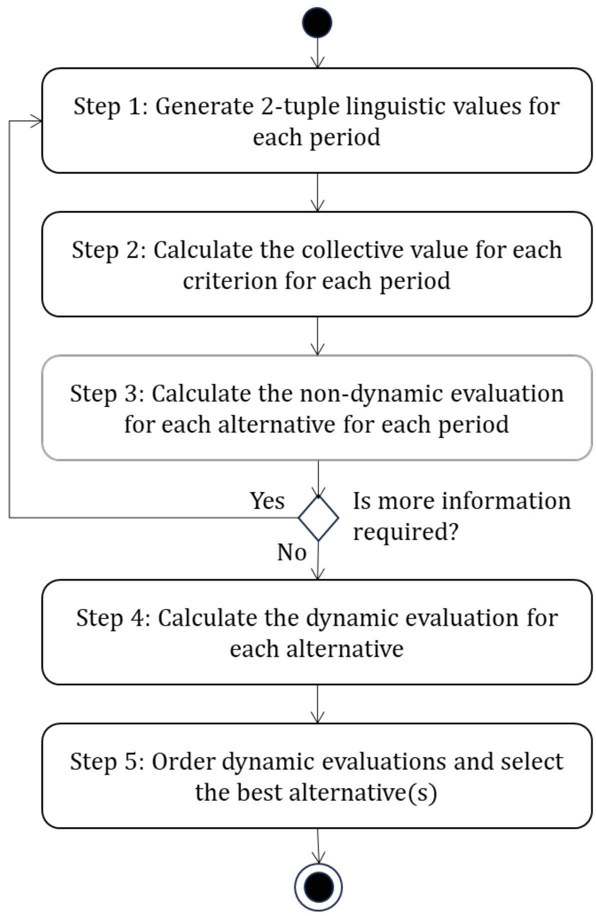

Figure 1. The flowchart of the proposed approach

5. Illustrative Example

In this section, we explore the applicability of the $2TDOWAWA_{V,W}$ operator extending an illustrative example presented in [14] to a 2-tuple linguistic context. A risk investment company wants to invest a sum of money in the best option. This problem involves the evaluation of four possible enterprises denoted as $A = \{a_1, a_2, a_3, a_4\}$. The attributes, $C = \{c_1, c_2, c_3, c_4\}$ are: c_1, the ability of sale, c_2, the ability of production, c_3, the ability of technology and c_4, the ability of financing. Three experts, $E = \{e_1, e_2, e_3\}$, provide assessment information on C in order to prioritize these enterprises A with respect to their performance. Weights of experts and criteria are assumed to be equal and constant over $T = \{t_1, t_2, t_3\}$.

In the following, we utilize the developed approach to select the best enterprise.

Step 1: Generate 2-tuple linguistic values for each period. Experts use the linguistic term set:

$$S = \{s_0 : Extremely\ Poor\ (EP), s_1 : Very\ Poor\ (VP), s_2 : Poor\ (P), s_3 : Medium\ (M), s_4 : Good\ (G), s_5 : Very\ Good\ (VG), s_6 : Extremely\ Good\ (EG)\},$$

to provide evaluation information for the enterprises in 2006–2008 according to the attributes and construct, respectively, the linguistic decision matrices $X(t_\lambda) = (x(t_\lambda))_{4\times 4}$.

The original linguistic information is listed in Tables 1-3. The 2-tuple linguistic values for each period are listed in Table 4, where t_1 denotes 2006, t_2 denotes 2007 and t_3 denotes 2008.

Table 1. x_{ijk} linguistic values for period t_1

e_k	a_i	c_1	c_2	c_3	c_4
	a_1	VP	P	G	M
	a_2	M	EP	M	VP
e_1	a_3	EP	VP	VG	G
	a_4	M	VG	VG	M
	a_1	G	P	M	G
	a_2	P	VP	P	VP
e_2	a_3	EP	EP	EG	P
	a_4	VG	G	M	G
	a_1	VP	VG	VG	P
	a_2	G	VP	VP	P
e_3	a_3	VP	M	G	G
	a_4	G	G	M	P

Step 2: To compute the collective value $\hat{X}(t_\lambda)$ for each criterion for each period for each alternative, we use the 2TAM aggregation operator from Definition 2 due to the experts being considered to be equally important.

Table 2. x_{ijk} linguistic values for period t_2

e_k	a_i	c_1	c_2	c_3	c_4
	a_1	M	G	VG	M
	a_2	P	VG	G	G
e_1	a_3	M	G	VG	G
	a_4	G	VG	G	M
	a_1	G	G	M	G
	a_2	G	M	M	G
e_2	a_3	G	VG	EG	P
	a_4	P	G	P	VG
	a_1	M	M	EG	EG
	a_2	G	M	VP	VG
e_3	a_3	P	G	VG	G
	a_4	M	G	M	M

Table 3. x_{ijk} linguistic values for period t_3

e_k	a_i	c_1	c_2	c_3	c_4
	a_1	VP	VG	G	M
	a_2	G	G	M	G
e_1	a_3	VP	P	VG	G
	a_4	M	G	P	VP
	a_1	P	P	EG	G
	a_2	VG	VG	G	VG
e_2	a_3	M	VP	EG	P
	a_4	P	M	M	G
	a_1	VP	VG	EG	EG
	a_2	G	VG	M	VG
e_3	a_3	VP	M	G	G
	a_4	G	P	P	M

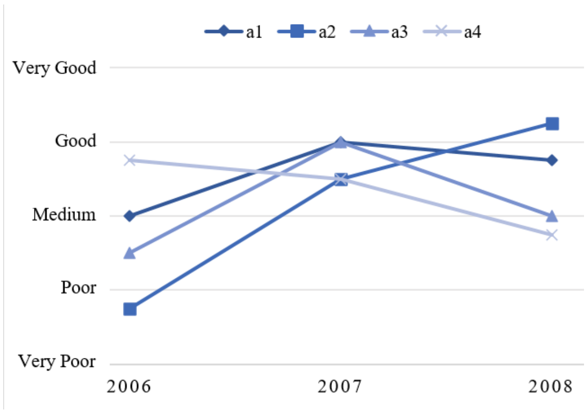

Figure 2. Non-dynamic 2-tuple linguistic evaluation of alternatives

Step 3: For computing the non-dynamic 2-tuple linguistic evaluation $\bar{X}(t_\lambda)$ for each alternative, we use the 2TAM aggregation operator in Eq. 4. Results are listed in Table 4 and depicted in Figure 2.

Table 4. $\bar{x}_i$ non-dynamic 2-tuple linguistic evaluation of alternatives for each period.

a_i	t_1	t_2	t_3
a_1	$(s_3, 0)$	$(s_4, 0)$	$(s_4, -0.25)$
a_2	$(s_2, -0.25)$	$(s_4, -0.50)$	$(s_4, 0.25)$
a_3	$(s_3, -0.50)$	$(s_4, 0)$	$(s_3, 0)$
a_4	$(s_4, -0.25)$	$(s_4, -0.50)$	$(s_3, -0.25)$

Step 4: Dynamic evaluations for each alternative are computed using different settings for the *2TDOWAWA*$_{V,W}$ as illustrated in Table 5. The weighting vectors are assumed to be $W = (0.5, 0.3, 0.2)$ and $V = (0.1, 0.3, 0.6)$. With V we can emphasize the higher importance of later evaluations.

Table 5. $\breve{X}(t_\lambda)$ outputs for different $\ell = 0$ values in the *2TDOWAWA*$_{V,W}$

ℓ	a_1	a_2	a_3	a_4
0	$(s_4, -.25)$	$(s_4, -.22)$	$(s_3, .25)$	$(s_3, .08)$
0.1	$(s_4, -.25)$	$(s_4, -.25)$	$(s_3, .27)$	$(s_3, .12)$
0.2	$(s_4, -.25)$	$(s_4, -.27)$	$(s_3, .28)$	$(s_3, .16)$
0.5	$(s_4, -.26)$	$(s_4, -.35)$	$(s_3, .33)$	$(s_3, .28)$
0.8	$(s_4, -.27)$	$(s_4, -.42)$	$(s_3, .37)$	$(s_3, .40)$
0.9	$(s_4, -.27)$	$(s_4, -.45)$	$(s_3, .39)$	$(s_3, .44)$
1.0	$(s_4, -.27)$	$(s_4, -.47)$	$(s_3, .40)$	$(s_3, .48)$
$2TDWA$	$(s4, -.25)$	$(s_4, -.22)$	$(s_3, .25)$	$(s_3, .08)$
$2TDOWA$	$(s_4, -.27)$	$(s_4, -.47)$	$(s_3, .40)$	$(s_3, .48)$
$2TDWG$ [23]	$(s4, -.26)$	$(s4, -.33)$	$(s3, .21)$	$(s3, .05)$
$2TDWHA$ [23]	$(s4, -.27)$	$(s4, -.48)$	$(s3, .17)$	$(s3, .03)$

Step 5: Orders of alternatives are listed in Table 6 per different ℓ values in the $2TDOWAWA_{V,W}$ operator.

Table 6. Dynamic aggregation operators, order and solution obtained

ℓ	Order
0	$a_2 \prec a_1 \prec a_3 \prec a_4$
0.1	$a_1 = a_2 \prec a_3 \prec a_4$
0.2	$a_1 \prec a_2 \prec a_3 \prec a_4$
0.5	$a_1 \prec a_2 \prec a_3 \prec a_4$
0.8	$a_1 \prec a_2 \prec a_4 \prec a_3$
0.9	$a_1 \prec a_2 \prec a_4 \prec a_3$
1.0	$a_1 \prec a_2 \prec a_4 \prec a_3$
$2TDWA$	$a_2 \prec a_1 \prec a_3 \prec a_4$
$2TDOWA$	$a_1 \prec a_2 \prec a_4 \prec a_3$
$2TDWG$ [23]	$a_1 \prec a_2 \prec a_3 \prec a_4$
$2TDWHA$ [23]	$a_1 \prec a_2 \prec a_3 \prec a_4$

The order of alternatives varies depending on the $2TDOWAWA_{V,W}$ class obtained by modulating the aggregation attitude with the ℓ value. The main advantage of the $2TDOWAWA_{V,W}$ operator is that it can provide different results regarding uncertainty according to the particular interests of the decision maker in the specific problem considered.

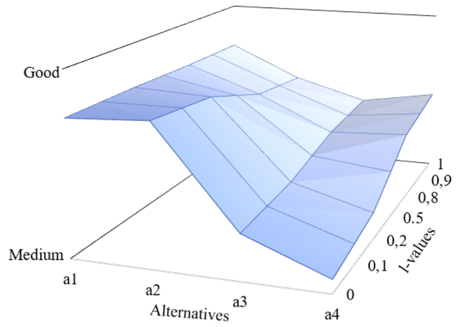

Figure 3. Dynamic 2-tuple linguistic evaluation of alternatives using different ℓ values

That is, if the experts consider the order of the evaluations without taking into account the periods, they can assign ℓ=0 and $2TDOWAWA_{V,W}$ will behave as the *2TDWA* aggregation operator, with a_2 being the best company. Otherwise, if they take into account only the importance of periods, they can assign ℓ=1 and $2TDOWAWA_{V,W}$ will behave as the *2TDOWA* aggregation operator; in this case a_1 would be the best company. Now, if the experts need to weight both the order of the values and the importance of the periods, the coefficient ℓ allows modeling that attitude, in such a way that it is possible to assign it values in the $[0, 1]$ interval whose midpoint (ℓ=0.5) would mean that both elements have the same weight in the decision; in that case company a_1 is better than company a_2. However, in this example it is clear that a_1 or a_2 is the optimal choice. In addition, we compare these results with those obtained using the 2-tuple dynamic weighted geometric (*2TDWG* [23]) and the 2-tuple dynamic weighted harmonic average (*2TDWHA* [23]) aggregation operators. As shown in Tables 5 and 6, the results are similar, with the particularity that operators *2TDWG* and *2TDWHA* produce a single fixed result, while $2TDOWAWA_{V,W}$ offers a range of options that allows modeling different attitudes in the problem solving process.

Figure 3 visualizes the behavior of the $2TDOWAWA_{V,W}$ operator for a set of ℓ values.

The proposed approach offers several advantages: The introduction of the ℓ parameter in the $2TDOWAWA_{V,W}$ provides decision-makers with a high degree of flexibility. They can adjust these parameters according to the specific requirements of the LDMADM problem.

This feature makes the 2-tuple LDMADM approach based on the $2TDOWAWA_{V,W}$ applicable to a wide range of real-world scenarios.

6. Conclusion

The selection of a suitable time-dependent 2-tuple linguistic aggregation operator is relevant due to its properties, which can highly modify the computing cost as well as results themselves and their accuracy and interpretability. The existing 2-tuple linguistic aggregation operators, such as $2TDA_{\phi}$, $2TDWA_{\theta}$ and $2TDOWA_{\vartheta}$ operators, can not simultaneously consider the information about the importance of the linguistic non-dynamic evaluation being aggregated and the importance of periods, and thus cannot equilibrate the influence of both kind of arguments on the final dynamic evaluation and the decision result. To solve this drawback, this paper introduced a new 2-tuple linguistic dynamic hybrid weighted aggregation operator, which is very useful to model different attitudes in decision-making by simultaneously weighting the given arguments as well as their ordered positions. The novel $2TDOWAWA_{V,W}$ weights not only the importance of a particular time period, but also the importance of non-dynamic evaluations in such a time period. Its main advantage is that it can unify the weighting behaviour of OWA and WA families by including the degree of importance of each concept in the aggregation. $2TDOWAWA_{V,W}$ is able to flexibly model situations where either the 2TDOWA or the 2TDWA fits the analysis. The parametric nature of the proposed $2TDOWAWA_{V,W}$ allows decision-makers to fine-tune the influence of time periods as well as non-dynamic evaluations at those periods. This level of control empowers decision-makers to precisely tailor the aggregation process to their preferences regarding the characteristics of the problem. Thus, the $2TDOWAWA_{V,W}$ is a more general approach to LDMADM.

In future work, we expect to develop further research on several families of $2TDOWAWA_{V,W}$ operators regarding the reordering process and the use of hybrid aggregation operators.

AUTHORS

Yeleny Zulueta-Véliz* – School of Computational Sciences and Technologies, Universidad de las Ciencias Informáticas, Havana, Cuba, https://orcid.org/0000-0003-0253-528X, e-mail: yeleny@uci.cu.

Carlos Rafael Rodríguez Rodríguez – Center for Project Management and Decision-Making, Universidad de las Ciencias Informáticas, Havana, Cuba, https://orcid.org/0000-0003-1803-0294, e-mail: crodriguezr@uci.cu.

Aylin Estrada Velazco – Department of Informatics, Universidad de las Ciencias Informáticas, Havana, Cuba, e-mail: avelazco@uci.cu, www: https://orcid.org/0000-0002-2557-361X.

*Corresponding author

ACKNOWLEDGEMENTS

The research presented in this paper was funded by the National Science Centre, Poland, under the grant no. 2019/35/O/ST6/01806.

References

[1] L. Zadeh, "Fuzzy Logic = Computing with Words," *Transactions on Fuzzy Systems*, vol. 4, no. 2, 1996, pp. 103–111, doi:10.1109/91.493904.

[2] L. Zadeh, "Fuzzy Logic—A Personal Perspective," *Fuzzy Sets and Systems*, vol. 281, 2015, pp. 4–20, doi:10.1016/j.fss.2015.05.009.

[3] L. Zadeh, "Fuzzy Sets," *Information and Control*, vol. 8, no. 3, 1965, pp. 338–353, doi:10.1016/S0019-9958(65)90241-X.

[4] Z. Pawlak, "Rough Sets," *International Journal of Computer & Information Sciences*, vol. 11, 1982, doi:10.1007/BF01001956.

[5] L. Zadeh, "Toward a Theory of Fuzzy Information Granulation and its Centrality in Human Reasoning and Fuzzy Logic," *Fuzzy Sets and Systems*, vol. 90, no. 2, 1997, pp. 111–127, doi:10.1016/S0165-0114(97)00077-8.

[6] H. Kamacı, S. Petchimuthu, and E. Akçetin, "Dynamic Aggregation Operators and Einstein Operations Based on Interval-Valued Picture Hesitant Fuzzy Information and their Applications in Multi-Period Decision Making," *Computational and Applied Mathematics*, vol. 40, 2021, doi:10.1007/s40314-021-01510-w.

[7] J. Morente-Molinera et al., "A Novel Multi-Criteria Group Decision-Making Method for Heterogeneous and Dynamic Contexts Using Multi-Granular Fuzzy Linguistic Modelling and Consensus Measures," *Information Fusion*, vol. 53, 2020, pp. 240–250, doi:10.1016/j.inffus.2019.06.028.

[8] R. Tao, et al., "A Dynamic Group MCDM Model with Intuitionistic Fuzzy Set: Perspective of Alternative Queuing Method," *Information Sciences*, vol. 555, 2021, pp. 85–103, doi:10.1016/j.ins.2020.12.033.

[9] F. Teng, C. Du, M. Shen, and P. Liu, "A Dynamic Large-Scale Multiple Attribute Group Decision-Making Method with Probabilistic Linguistic Term Sets Based on Trust Relationship and Opinion Correlation," *Information Sciences*, vol. 612, 2022, pp. 257–295, doi:10.1016/j.ins.2022.07.092.

[10] Y. Zulueta-Veliz and L. García-Cabrera, "A Choquet Integral-Based Approach to Multiattribute Decision-Making with Correlated Periods," *Granular Computing*, vol. 3, no. 3, 2018, pp. 245–256, doi:10.1007/s41066-018-0095-4.

[11] F. Herrera and L. Martínez, "A 2-tuple Fuzzy Linguistic Representation Model for Computing with Words," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 6, 2000, pp. 746–752.

[12] T. Malhotra and A. Gupta, "A Systematic Review of Developments in the 2-tuple Linguistic Model and its Applications in Decision Analysis," *Soft Computing*, vol. 27, 2020, doi:10.1007/s00500-020-05031-2.

[13] Y. Zulueta-Veliz and P. J. Sanchez, "Linguistic Dynamic Multicriteria Decision Making Using Symbolic Linguistic Computing Models," Granular Computing, vol. 3, no. 3, 2018, pp. 229–244, doi:10.1007/s41066-018-0088-3.

[14] Z. Xu, "Multi-Period Multi-Attribute Group Decision-Making Under Linguistic Assessments," *International Journal of General Systems*, vol. 38, no. 8, 2009, pp. 823–850, doi:10.1080/03081070903257920.

[15] Y. Liu, "A Method for 2-tuple Linguistic Dynamic Multiple Attribute Decision Making with Entropy Weight," *Journal of Intelligent Fuzzy Systems*, vol. 27, no. 4, 2014, pp. 1803–1810, doi:10.3233/IFS-141147.

[16] R. Yager, "On Ordered Weighted Averaging Aggregation Operators in Multicriteria Decision Making," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. I, no. 18, 1988, pp. 183–190, doi:10.1109/21.87068.

[17] D. García-Zamora et al., "Admissible OWA Operators for Fuzzy Numbers," *Fuzzy Sets and Systems*, vol. 480, 2024, doi:10.1016/j.fss.2024.108863.

[18] V. Torra, "The Weighted OWA Operator," *International Journal of Intelligent Systems*, vol. 12, 1997, pp. 153–166.

[19] Z. Xu and Q. Da, "An Overview of Operators for Aggregating Information," *International Journal of Intelligent Systems*, vol. 18, 2003, pp. 953–969.

[20] J. Merigo, "On the Use of the OWA Operator in the Weighted Average and its Application in Decision Making," *Lecture Notes in Engineering and Computer Science*, vol. 41, 2009.

[21] L. Martínez, R. M. Rodríguez, and F. Herrera, *The 2tuple Linguistic Model: Computing with Words in Decision Making*, Springer International Publishing, 2015, p. 162, doi:10.1007/978-3-319-24714-4.

[22] G. Jimenez and Y. Zulueta, "A 2-tuple Linguistic Multi-Period Decision Making Approach for Dynamic Green Supplier Selection," *Revista DYNA*, vol. 84, no. 202, 2017, pp. 199–206, doi:10.15446/dyna.v84n202.58032.

[23] F. Ai and J. Yang, "Approaches to Dynamic Multiple Attribute Decision Making with 2-tuple Linguistic Information," *Journal of Intelligent Fuzzy Systems*, vol. 27, no. 6, 2014, pp. 2715–2723, doi:10.3233/IFS-131094.